Hyperelliptic Hodge Integrals

$$\int_{\overline{\mathcal{H}}_{g,2g+2}} (\psi_j)^{2g-1-i} \lambda_i \;= \;?$$

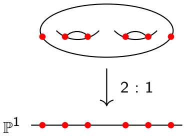
$\mathbb{L}_j$ $\longrightarrow$ $\overline{\mathcal{H}}_{g,2g+2}$ $\longleftarrow$ $\mathbb{E}_g$

$$\psi_j := c_1(\mathbb{L}_j) \qquad \lambda_i := c_i(\mathbb{E}_g)$$

$$\psi_j \in A^1(\overline{\mathcal{H}}_{g,2g+2}) \qquad \lambda_i \in A^i(\overline{\mathcal{H}}_{g,2g+2})$$

$\overline{\mathcal{H}}_{g,2g+2}$

$\cup$



Theorem (A)

$$\int_{\overline{\mathcal{H}}_{g,2g+2}} (\psi_j)^{2g-1-i} \lambda_i = \left(\frac{1}{2}\right)^{i+1} e_i(1, 3, 5, \ldots, 2g-1)$$

As you insert more λ-classes into the integrand, these intersection numbers exhibit more complicated combinatorial phenomena. But that is a story for another time.

Identities in prime rings

$$\sum_{i,j=0}^{n} \lambda_{ij} a^i x a^j = 0, \text{ for all } x \in R$$

a* algebraic with minimal polynomial *p iff

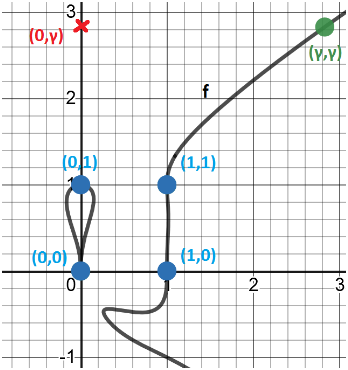
$$f(X,Y) := \sum_{i,j=0}^{n} \lambda_{ij} X^i Y^j \in \langle p(X), p(Y) \rangle$$

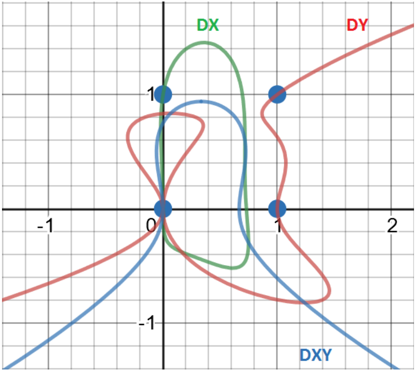
Main Theorem

If $p := \prod_{i=1}^{m}(X-\lambda_i)^{e_i}$ then $f \in \langle p(X), p(Y)\rangle$ iff

$$D_{X^r Y^s} f(\lambda_i, \lambda_j) = 0$$

for all λ_i, λ_j and all $0 \leq r < e_i, 0 \leq s < e_j$.





- $X^2(X-1)$ is a possible minimal polynomial, $X^2(X-1)^2$ is not

Computing Syzygies

Juliette Bruce

Where I work and live occupies the ancestral and current home land of the Muwekma Ohlone Tribe and other familial descendents of the sovereign Verona Band of Alameda County.

Black and Indigenous Lives Matter.

Syzygies & Algebraic Varieties

- A syzygy is a polynomial relation among a collection of polynomials.

- <u>Ex</u>: Consider the monomials of degree 3 on s and t: s^3, s^2t, st^2, t^3.

  - <u>0th Syzygies</u>

$$(s^3)(st^2) - (s^2t)^2 = 0$$

$$(s^2t)(t^3) - (st^2)^2 = 0$$

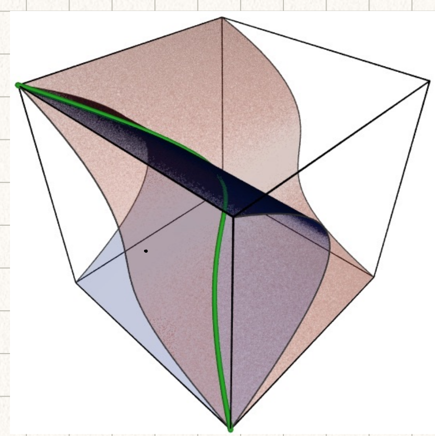
$$(s^3)(t^3) - (s^2t)(st^2) = 0$$

$x = s^3$, $y = s^2t$
$z = st^2$, $w = t^3$

F_1: $xz - y^2 = 0$

F_2: $yw - z^2 = 0$

F_3: $xw - yz = 0$



  - <u>1st Syzygies</u>

$$z \cdot F_1 + x \cdot F_2 - y \cdot F_3 = 0$$

$$w \cdot F_1 + y \cdot F_2 - z \cdot F_3 = 0$$

there are no relations among these

Veronese Syzygies (n=3)

- The standard approach using Gröbner bases bogs down when $n=3$ and $d=5$, but we can convert the question of computing syzygies into linear algebra.

Via Koszul cohomology techniques

- Namely we need compute the ranks of a number of really really really large matrices...

$$254,103,788,400 \times 902,737,143,000$$

- Using symmetries (multigradings) these huge matrices can be broken into a large number of very big matrices.

↑ Part of the $d=6$, $n=3$ computation breaks down to 1,753 matrices w/ the largest being

$$4,175,947 \times 12,168,528$$

SyzygyData.com

Multistationarity in chemical reaction networks

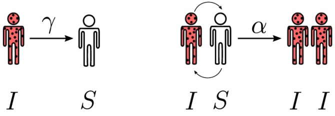
Laura Brustenga i Moncusí

LauraBMo/CRNT.jl

17th February 2021

KØBENHAVNS UNIVERSITET

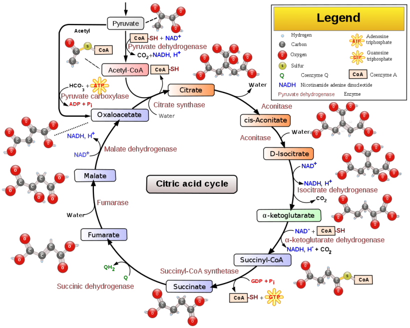
Reaction networks



SIS epidemiological model

$$2\,H + O_2 \longrightarrow 2\,H_2O$$
$$C + O_2 \longrightarrow CO_2$$

Combustion

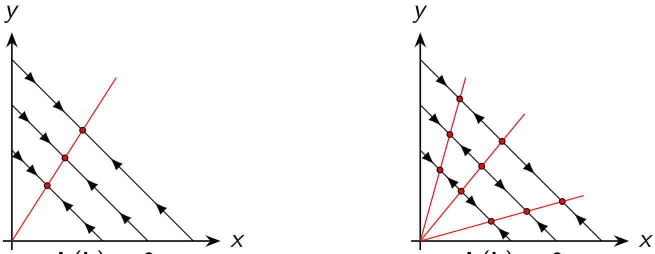


Krebs' cycle

The model: Mass-action law

$$3\,\mathrm{Y} \rightleftharpoons 2\,\mathrm{Y} + \mathrm{X}$$
$$\mathrm{Y} + 2\,\mathrm{X} \rightleftharpoons 3\,\mathrm{X}$$

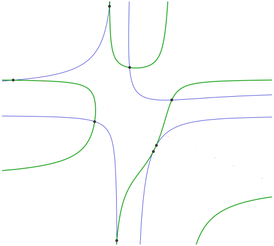
$$\begin{aligned}\dot{x} &= \quad k_1 y^3 - k_2 y^2 x + k_3 y x^2 - k_4 x^3 \\ \dot{y} &= -k_1 y^3 + k_2 y^2 x - k_3 y x^2 + k_4 x^3\end{aligned}$$



$\Delta(\mathbf{k}) < 0$ $\qquad$ $\Delta(\mathbf{k}) > 0$

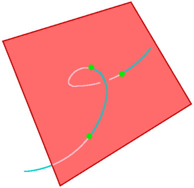
```
using Nemo # polynomial manipulations, exact operations,...
using Polymake # cones intersection, outer normal cone,...
using PolynomialRoots # find k's with some properties
```

Taylor Brysiewicz - Numerical Algebraic Geometry



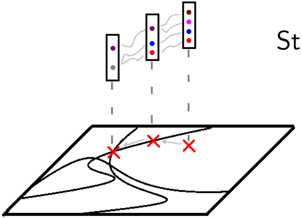
Solving systems

- Solve 0-dim systems by approximating solutions
- Can prove approximate sols $\leftrightarrow$ actual sols
- Can prove approximate sols $\leftrightarrow$ real sols



Positive dim'l varieties

- Study positive dim'l varieties via slices
- Irreducible decomposition, multiplicity, local dimension, etc...
- Projections are easy



Study parameter spaces

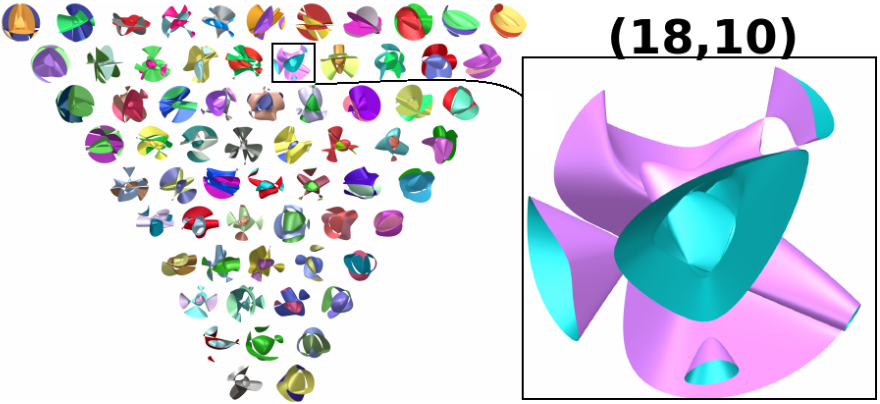
- Great for enumerative problems (Schubert, sparse, etc)
- Understand special fibres
- Monodromy groups
- Search for fibres with certain properties

Classification of quintic spectrahedra

Combinatorial type a spectrahedron:
(#real singularities on $\overline{\partial X}$, #real singularities on ∂X)

Theorem (Brysiewicz, Kozhasov, Kummer 2020)

There are 65 combinatorial types of quintic spectrahedra.



Witnesses for each of the 65 different combinatorial types of quintic spectrahedra.

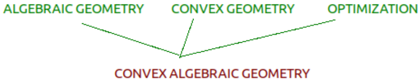
Combinatorial Aspects of Convex Algebraic Geometry

Papri Dey

Department of Mathematics
University of Missouri, Columbia

ICERM Fall 2021 CAG:February 17,
Lightning Talk

University of Missouri, Columbia



Main objects: compute the convex hull of a given algebraic variety or semialgebraic set

"Semidefinite Optimization and Convex Algebraic geometry ". - [BPT]

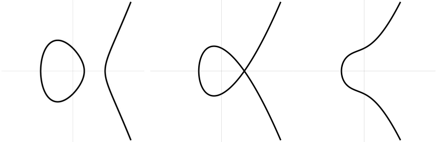
1. SPECTRAHEDRA - feasible sets of semidefinite programming
2. DETERMINANTAL REPRESENTATION PROBLEMS
3. HYPERBOLIC POLYNOMIALS

Definition

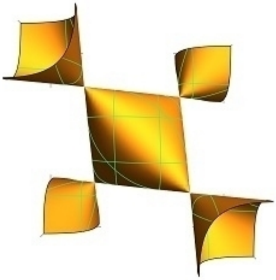
A form $F \in \mathbb{R}[x_0, x_1, \ldots, x_n]$ is hyperbolic w.r.t $e \in \mathbb{R}^{n+1}$ if

1. $F(e) \neq 0$
2. $F(a + te) \in \mathbb{R}[t]$ is *real rooted* for all $a \in \mathbb{R}^{n+1}$

Consider the family of cubic forms $F_c = zy^2 - (y - \frac{1}{c}z)(y^2 - cz^2)$ where $c \in \mathbb{R} \setminus \{0\}$.



1. Determinantal polynomials: $f(\mathbf{x}) = \det(\sum_{i=0}^{n} x_i A_i) \in \mathbb{R}[\mathbf{x}]$ where A_i's are symmetric or Hermitian matrices
2. Symmetroids: $\mathcal{V}(f)$, zero set of the determinant of a symmetric linear matrix form
3. Spectrahedra: $S(f) := \{\mathbf{x} \in \mathbb{RP}^n : \sum_{i=0}^{n} x_i A_i \text{ is semidefinite}\}$.



x1x2x3+x1x2x4+x1x3x4+x2x3x4

$$= \begin{vmatrix} x1+x4 & x4 & x4 \\ x4 & x2+x4 & x4 \\ x4 & x4 & x3+x4 \end{vmatrix}$$

CAYLEY SYMMETROID

HELTON-VINNIKOV THEOREM

LAX CONJECTURE

SPECTRAHEDRA

HYPERBOLICITY CONE

GENERALIZED LAX CONJ.

SEMIDEFINITE PROGRAMMING

DeterminantalRepresentations.m2 [CD]

A generalization of Springer's representations in type A (and Sage!)

Sean Griffin

ICERM → UC Davis

Current Research: Representations of S_n coming from geometry.

- Dimension formulas
- Decomposition into irreducibles (Specht modules S^λ)

Background

Springer Fibers (Type A):

A family of subvarieties $\mathcal{B}^\lambda \subseteq Fl(n)$
(λ a partition of n)

Ppt $S_n \circlearrowright H^*(\mathcal{B}^\lambda)$ even though
$S_n \not\circlearrowright \mathcal{B}^\lambda$.

Thm (Springer) Geom. construction of S^λ:

$$H^{top}(\mathcal{B}^\lambda) \cong S^\lambda$$

Thm (Spaltenstein)

$$\left\{ \begin{matrix} \text{Irreducible} \\ \text{Components of } \mathcal{B}^\lambda \end{matrix} \right\} \longleftrightarrow SYT(\lambda)$$

Other connections: Graded multiplicities of S^μ in $H^*(\mathcal{B}^\lambda)$ given by q-Kostka poly.

Current Work

We introduce a new family of varieties $Y_{n,\lambda}$ where λ has size $\leq n$.

When λ has size n, $Y_{n,\lambda} = \mathcal{B}^\lambda$.

Thm (G.-Levinson-Woo)

$$H^{top}(Y_{n,\lambda}) \cong \mathrm{Ind}\big\uparrow_{S_k}^{S_n} (S^\lambda).$$

Thm (G.-Levinson-Woo)

$$\left\{ \begin{matrix} \text{Irred. cmpt.} \\ \text{of } Y_{n,\lambda} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Injective} \\ \text{tableaux on } \lambda \end{matrix} \right\}$$

Other: Formulas for graded S_n-character of $H^*(Y_{n,\lambda})$.

Connections to the Delta Conjecture from Algebraic Combinatorics.

Key Tools:

- An explicit presentation of $H^*(Y_{n,\lambda})$:

$$H^*(Y_{n,\lambda}) \cong \frac{\mathbb{Z}[x_1,\ldots,x_n]}{I_{n,\lambda}}.$$

- Frobenius characteristic map

$$\underset{\text{(Groth. group of)}}{S_n\text{-reps}} \xrightarrow[\sim]{\text{Frob}} \text{Symmetric functions of deg. } n$$

Many graded S_n-reps in combinatorics are quotient rings, e.g.

Diagonal coinvariants, Zabrocki's Superspace, parking space, ...

Project: Write Sagemath code to compute Frob of a symmetric graded ring from a presentation of the ring.

On the deformation rigidity of smooth projective symmetric varieties with Picard number one

Shin-Young Kim (IBS-CGP)
joint work with Kyeong-Dong Park

Sage/Oscar Days for Combinatorial Algebraic Geometry
Feb 15 - 19, 2021, Virtual Conference

Theorem (2010 A.Ruzzi)

Let S be a smooth projective completion of a symmetric space G/G^θ with Picard number one. If S is non-homogeneous, then the restricted root system is either G_2 or A_2. Moreover, S is a section of a homogeneous space.

Combining this with geometric Freudenthal-Tits magic square,

	G/G^θ	S	$\mathcal{C}(S)$
G_2	$G_2/(\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C}))$	$\mathrm{Gr}(3,7)\cap L^{27}\subset\mathbb{P}^{34}$	$\mathbb{P}^1\times\mathbb{P}^1$
	$G_2\times G_2/G_2$	$\mathrm{Gr}_\omega(7,14)\cap L^{46}\subset\mathbb{P}^{49}$	G_2/P_{α_1}
A_2	$\mathrm{SL}(3,\mathbb{C})/\mathrm{SO}(3,\mathbb{C})$	$\mathrm{LGr}(3,6)\cap H$	$v_4(\mathbb{P}^1)$
	$\mathrm{SL}(3,\mathbb{C})\times\mathrm{SL}(3,\mathbb{C})/\mathrm{SL}(3,\mathbb{C})$	$\mathrm{Gr}(3,6)\cap H$	$\mathbb{P}(T_{\mathbb{P}^2})$
	$\mathrm{SL}(6,\mathbb{C})/\mathrm{Sp}(6,\mathbb{C})$	$\mathbb{S}_6\cap H$	$\mathrm{Gr}_\omega(2,6)$
	E_6/F_4	$E_7/P_7\cap H$	$\mathbb{OP}^2_0$

where H is a general hyperplane and L^k is a linear subspace with dimension k.

Theorem ('19, K.- and Park)

Let $\pi\colon \mathcal{X} \to \Delta$ be a smooth projective morphism from a complex manifold $\mathcal{X}$ to the unit disc $\Delta \subset \mathbb{C}$. Denote by S the smooth equivariant completion with Picard number one of the symmetric homogeneous space $\mathrm{SL}(6, \mathbb{C})/\,\mathrm{Sp}(6, \mathbb{C})$ or E_6/F_4.

Suppose for any $t \neq 0$, the fiber $\mathcal{X}_t = \pi^{-1}(t)$ is biholomorphic to the smooth projective symmetric variety S. Then the central fiber $\mathcal{X}_0$ is biholomorphic to either S or an equivariant compactification of the vector group $\mathbb{C}^n$, $n = \dim S$.