Computing the Newton polytope of a large discriminant

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Sage/Oscar Days for Combinatorial Algebraic Geometry

- Joint work with Robert Löwe
- – and Robert Löwe. The Newton polytope of the discriminant of a quaternary cubic form (https://arxiv.org/abs/1909.08910)
- Charles Jordan, Michael Joswig, –: Parallel Enumeration of Triangulations. Electron. J. Comb. 2018 (www.combinatorics.org/ojs/index.php/eljc/article/view/v25i3p6)

Large computer experiments

Parallelization in computer algebra

- `MPTOPCOM` for parallel computation of triangulations (accessible from `polymake` and `Polymake.jl`)
- `Singular` and GPI-Space (talk by Anne Frühbis-Krüger on Thursday)
- `mplrs` for computing convex hulls in parallel
- New framework for computing the fan induced by a hyperplane arrangement in parallel

Running these on a cluster one often encounters similar problems. It is good to have these in mind when implementing new software.

Large computer experiments

What characterises a large computer experiment?

- Long runtime (80 days)
- Many parallel processes (128 slots)
- Output is very large (338.2GiB, 16.5GiB compressed)

Main problems:

- Can often only be run once
- Hard to verify (especially for referees)

Need to be carefully planned and documented!

Large computer experiments

Documentation of computer experiments

- ICERM Workshop on Reproducibility in Computational and Experimental Mathematics, organized by V. Stodden, D. H. Bailey, J. Borwein, R. J. LeVeque, W. Rider, and W. Stein
- Write clean code

Today

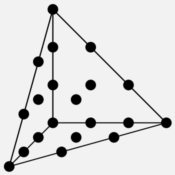
- Requirements on the software and algorithms
- How does computation on a cluster work

Mathematical background

Quaternary cubic form

$$
\begin{aligned}
f = \; & a_{(3,0,0,0)} \cdot w^3 + a_{(2,0,0,1)} \cdot w^2 z^1 + a_{(1,0,0,2)} \cdot w^1 z^2 + a_{(0,0,0,3)} \cdot z^3 + \\
& a_{(2,0,1,0)} \cdot w^2 y^1 + a_{(1,0,1,1)} \cdot w^1 y^1 z^1 + a_{(0,0,1,2)} \cdot y^1 z^2 + a_{(1,0,2,0)} \cdot w^1 y^2 + \\
& a_{(0,0,2,1)} \cdot y^2 z^1 + a_{(0,0,3,0)} \cdot y^3 + a_{(2,1,0,0)} \cdot w^2 x^1 + a_{(1,1,0,1)} \cdot w^1 x^1 z^1 + \\
& a_{(0,1,0,2)} \cdot x^1 z^2 + a_{(1,1,1,0)} \cdot w^1 x^1 y^1 + a_{(0,1,1,1)} \cdot x^1 y^1 z^1 + a_{(0,1,2,0)} \cdot x^1 y^2 + \\
& a_{(1,2,0,0)} \cdot w^1 x^2 + a_{(0,2,0,1)} \cdot x^2 z^1 + a_{(0,2,1,0)} \cdot x^2 y^1 + a_{(0,3,0,0)} \cdot x^3
\end{aligned}
$$

- **cubic**: Homogeneous of degree 3.
- **quaternary**: Four variables.
- f defines a surface $V(f)$ in $\mathbb{P}^3$.
- The exponent vectors of f are the lattice points of $3\Delta_3$.



Question: When is $V(f)$ singular?

Answer: Plug the coefficients of f into the discriminant $\mathcal{D}$. If it gives zero, then $V(f)$ has singularities.

Mathematical background

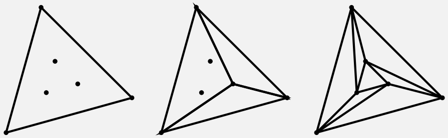
Problem

The only things we know about the discriminant $\mathcal{D}$ of quaternary cubic forms are

- Polynomial in 20 variables ($=$ number of coefficients of f)
- Homogeneous of degree 32.

Maybe we can determine the Newton polytope of $\mathcal{D}$?

Given a regular triangulation of $3 \cdot \Delta^3$, its D-equivalence class is the exponent vector of an extremal monomial ($=$ vertex of the Newton poloytope) of $\mathcal{D}$. All extremal monomials arise in this way.



Mathematical background

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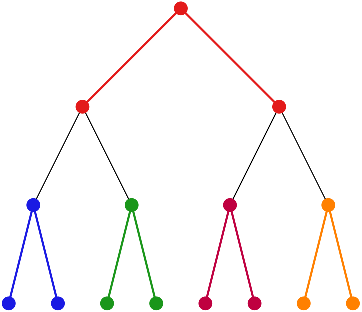
Task

Find all regular triangulations of $3 \cdot \Delta^3$. (20 points, symmetry group of size 24)
We know: $3 \cdot \Delta^3$ has 21 125 102 regular full triangulations.

Reverse search



- parallel version:
  budgeted reverse search
- depth first search, but subtrees get redistributed when budget is exhausted
- budget can be maximum depth allowed
- no communication between workers
- one separate master for redistributing jobs
- one output worker

Setup

Software used

- `MPTOPCOM` v1.0
- `polymake` v4.0
- `OpenMPI` v3.1.3
- `mts` for parallel reverse search
- `soplex` for checking regularity of triangulations
- Scientific Linux

Planning

- Try software setup on smaller examples first
  - Test that the software gives the correct results
  - Test that your code does the right thing (e.g. reads and writes the correct files)
- Use well tested software

Running on a cluster

Scheduler

- Scheduling system that organizes jobs
- Jobs have to specify the amount of resources they need (#slots, memory, etc., runtime)
- Jobs running out of resources are killed
- Jobs with low resource requirements are more likely to fit into gaps left by larger jobs (for us: many slots, but short runtime worked well)

Checkpointing

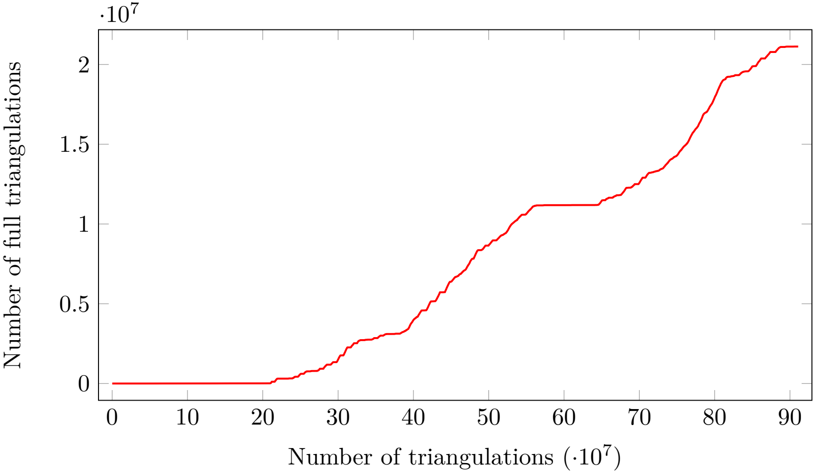
- Continue computation from a checkpoint. (Short 12h jobs, 128 slots)
- Subdivide computation into jobs
- Deal with unexpected events, like power outages
- Intermediate results can help estimate the remaining runtime
- Subresults can help estimate the remaining runtime

Running on a cluster

Requirements on software

1. Checkpointing: Splitting the computation up into jobs
2. Estimate resource usage
3. Can you estimate the runtime from the intermediate results? (Boehm, Joswig, –, Newman: Random growth on a Ramanujan graph)
4. Metadata for verifying intermediate results (e.g. how many triangulations were counted vs how many are actually written on disk)
5. Devise method for repairing data between checkpoints

Estimating runtime



Data homepage

dequivalence

Regular triangulations of the 3-dilated 3-simplex

Table of Contents

Download

You can find all regular triangulations of the 3-dilated 3-simplex here. There are 910,974,879 triangulations in total.

Software used

These output files were produced by mptopcom v1.0. The triangulations are denoted up to group action by the linear symmetry group on the simplex.

Data format

After extracting, every line in the output files corresponds to a triangulation orbit. E.g.:

Our experiment

Results

- $3\Delta^3$ has 910 974 879 regular triangulations.
- There are 166 104 D-equivalence classes.
- 80 days on a 128 core cluster.
- 189 chunks from 189 12-hour jobs.
- 7 years on a 4-core laptop

The end

Thank you for your attention.

1. – and Robert Löwe. The Newton polytope of the discriminant of a quaternary cubic form (https://arxiv.org/abs/1909.08910)
2. Download our data at https://polymake.org/doku.php/dequivalence
3. polymake: https://polymake.org
4. Charles Jordan, Michael Joswig, –: Parallel Enumeration of Triangulations. Electron. J. Comb. 2018 (www.combinatorics.org/ojs/index.php/eljc/article/view/v25i3p6)
5. MPTOPCOM: https://polymake.org/mptopcom
6. Janko Boehm, Michael Joswig, –, Andrew Newman: Random growth on a Ramanujan graph (https://arxiv.org/abs/1908.09575)
7. Janko Boehm, Anne Frühbis-Krüger, Mirko Rahn: Massively parallel computations in algebraic geometry - not a contradiction (https://arxiv.org/abs/1811.06092)
8. Dominik Bendle, Janko Boehm, Yue Ren, Benjamin Schröter: Parallel Computation of tropical varieties, their positive part, and tropical Grassmannians (https://arxiv.org/abs/2003.13752)