Friday, August 13, 2021
2:00 – 3:00 EDT
Problem Session

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Question

k field of prime char $p > 0$

$X = (x_{ij})$ generic $n \times m$ matrix

$$R = \frac{k[X]}{I_t(X)}, \quad \mathfrak{m} = (x_{ij} \mid 1 \le i, j \le n)$$

Find an explicit $C > 0$ such that

$$P^{(Cn)} \subseteq \mathfrak{m}^n$$

for all $n \geq 1$, all homogeneous primes P

Possible guess: $C = 2$

Why? Because that's the answer in char 0 (!)

If char $k = 0$,

$X = (x_{ij})$ generic $n \times m$ matrix

$$R = \frac{k[X]}{I_t(X)}, \quad \mathfrak{m} = (x_{ij} \mid 1 \le i, j \le n)$$

We know $P^{(2n)} \subseteq \mathfrak{m}^n$ for all $n \geq 1$,

because:

- We saw in Anurag's talk that in this case,

$R \cong k[YZ] \overset{\oplus}{\hookrightarrow} k[Y, Z]$ where

Y is an $m \times (t-1)$ matrix
Z is an $(t-1) \times n$ matrix
$\Rightarrow R$ generated in deg 2

- (Dao – De Stefani – G – Huneke – Núñez Betancourt) (*)

k field

$R = k[f_1, \dots, f_\ell] \overset{\oplus}{\hookrightarrow} S = k[x_1, \dots, x_d]$

$f_1, \dots, f_\ell$ homogeneous, $D := \max_i \deg(f_i)$

then

$$Q^{(Dn)} \subseteq \mathfrak{m}^n$$

for all homogeneous primes Q and all $n \geq 1$

<u>Special case</u>:

<u>Theorem</u> (Carvajal-Rojas — Smolkin, 2020)

char $k = p > 0$

$$R = \frac{k[a, b, c, d]}{(ad - bc)} \cong \frac{k[X_{2\times 2}]}{I_2(X)}$$

then $\mathfrak{P}^{(hn)} \subseteq \mathfrak{P}^n$ for all $n \geq 1$, $\mathfrak{P}$ prime of height h.
hom

$\Rightarrow \mathfrak{P}^{(2n)} \subseteq \mathfrak{P}^n \subseteq \mathfrak{m}^n \Rightarrow C = 2$ works

Cannot do better:

$I = (a, c, d)$ $\quad \underbrace{b}_{\notin I} c = ad \in I^2 \Rightarrow c \in I^{(2)} \setminus m^2$

$\Rightarrow C = 1$ won't work for all primes, so $C = 2$ is best possible

Notice however that in this case, R is a direct summand of a polynomial ring:

$$R = \frac{k[a,b,c,d]}{(ad-bc)} \cong k[xy, xu, yv, uv] \hookrightarrow k[x,y,u,v]$$

so the fact that we can take $C = 2$ also follows from (*).

Follow-up problems:

k field of characteristic $p > 0$

2) Symmetric determinantal rings:

$k[X] / I_t(X)$, $\quad X$ generic $n \times n$ symmetric matrix

3) Pfaffian determinantal rings:

$k[X] / Pf_{2t}(X)$ $\quad X$ $n \times n$ generic alternating matrix

STUDYING SINGULARITIES USING CLOSURE OPERATIONS

REBECCA R.G.

Background:
Recall that for an R-module M, the trace ideal of M is

$$\operatorname{tr}_M(R) = \sum_{f \in \operatorname{Hom}_R(M,R)} f(M).$$

When M is finitely-generated, this agrees with the test ideal of the closure operation cl_M.
Problem:
Compute the test/trace ideals of some finitely-generated Cohen-Macaulay modules (also called maximal Cohen-Macaulay modules). The examples given below come from Example 5.25 in Cohen-Macaulay Representations by Leuschke and Wiegand.

(1) Let $R = k[[u^5, u^2v, uv^3, v^5]] \subseteq k[[u,v]] = S$, where k has characteristic not equal to 5. The indecomposable finitely-generated Cohen-Macaulay R-modules are:
  (a) $M_0 = R$,
  (b) $M_1 = R(u^4, uv, v^3) \cong (u^5, u^2v, uv^3)$,
  (c) $M_2 = R(u^3, v) \cong (u^5, u^2v)$,
  (d) $M_3 = R(u^2, uv^2, v^4) \cong (u^5, u^4v^2, u^3v^4)$, and
  (e) $M_4 = R(u, v^2) \cong (u^5, u^4v^2)$.
  Compute the trace ideals of these modules in Macaulay2 or by hand. What is the intersection of all of the trace modules?
(2) Let $R = k[[u^8, u^3v, uv^3, v^8]] \subseteq k[[u,v]] = S$, where k has characteristic not equal to 2. The indecomposable finitely-generated Cohen-Macaulay R-modules are:
  (a) $M_0 = R$,
  (b) $M_1 = R(u^7, u^2v, v^3) \cong (u^8, u^3v, uv^3)$,
  (c) $M_2 = R(u^6, uv, v^6) \cong (u^8, u^3v, u^2v^6)$,
  (d) $M_3 = R(u^5, v) \cong (u^8, u^3v)$,
  (e) $M_4 = R(u^4, u^2v^2, v^4) \cong (u^8, u^6v^2, u^4v^4)$,
  (f) $M_5 = R(u^3, uv^2, v^7) \cong (u^8, u^6v^2, u^5v^7)$,
  (g) $M_6 = R(u^2, u^5v, v^2) \cong (u^2v^6, u^5v^7, v^8)$, and
  (h) $M_7 = R(u, v^5) \cong (uv^3, v^8)$.
  Compute the trace ideals of these modules in Macaulay2 or by hand. What is the intersection of all of the trace modules?
(3) Both of these were examples of a more general set of rings. Let $S = k[[u,v]]$, $r \geq 2$ an integer not divisible by $\operatorname{char}(k)$, and choose $0 < q < r$ with $(q,r) = 1$. Take

$G = \langle g \rangle \cong \mathbb{Z}/r\mathbb{Z}$ to be the cyclic group of order r generated by $g = \begin{pmatrix} \xi_r & 0 \\ 0 & \xi_r^q \end{pmatrix} \in GL(2, k)$, where ξ_r is a primitive rth root of unity.

Let $R = k[[u, v]]^G$ be the corresponding ring of invariants, so that R is generated by the monomials $u^a v^b$ satisfying $a + bq \equiv 0 \pmod{r}$. The indecomposable finitely-generated Cohen-Macaulay R-modules are

$$M_j = R(u^a v^b \mid a + qb \equiv -j \pmod{r}).$$

(a) Confirm that this gives rise to the first example when $r = 5$ and $q = 3$ and the second example when $r = 8$ and $q = 5$.

(b) For an arbitrary q and r, what are the test/trace ideals of the M_j? What is the intersection of the trace ideals of all of the M_j? You can start with the case $r = q + 1$ (this is done in Benali-Pothagoni-R.G., but via an isomorphism to an (A_q) hypersurface singularity).

(4) A more general open question is: how do the test/trace ideals of finitely-generated Cohen-Macaulay modules correspond to the singularities of the ring? Previous work of the speaker (Perez-R.G.) and of Herzog-Hibi-Stamate shows some cases where there are connections. Choose a class of finitely-generated Cohen-Macaulay modules and compute their trace ideals. Do certain properties of these trace ideals correspond to existing classes of singularities? Or can you find a new class of singularities corresponding to particular behavior of finitely-generated Cohen-Macaulay module trace ideals?

Relevant Reading:

Skim through `https://arxiv.org/abs/2103.02529` (Benali-Pothagoni-R.G.). Pay particular attention to Definition 2.10, how they compute examples of test/trace ideals, and how the examples relate to the properties of the rings.

The theoretical basis for this research approach is developed in `https://arxiv.org/abs/1907.02150` (Perez-R.G.).

More examples of finitely-generated Cohen-Macaulay modules (including all the ones appearing in Benali-Pothagoni-R.G.) can be found in Cohen-Macaulay Representations by Leuschke and Wiegand, and Cohen-Macaulay Modules over Cohen-Macaulay Rings by Yoshino.

Work on trace ideals of canonical modules appears in these papers (2 of them by Herzog-Hibi-Stamate): `https://arxiv.org/search/?query=herzog+hibi+stamate&searchtype=all&source=header`.

Open problems

Thomas Reichelt

All problems presented deal with GKZ systems M_A^0 where A is a $d \times n$ integer matrix which satisfies

1. $\mathbb{Z}A = \mathbb{Z}^d$
2. A is **saturated**, i.e. $\mathbb{R}_{\geq 0}A \cap \mathbb{Z}^d = \mathbb{N}A$
3. A is **pointed**: $\mathbb{N}A \cap (-\mathbb{N}A) = \{0\}$.

Question 1: What is the length of M_A^0?

Let σ be the cone $\mathbb{R}_{\geq 0}A$, i.e. the cone generated by the columns of A. Denote by $\sigma^\vee$ the dual cone and by $\gamma^\perp$ the annihilator of γ. There is a containment reversing bijection

$$\gamma \quad \leftrightarrow \quad \gamma^* = \gamma^\perp \cap \sigma^\vee$$

If γ has dimension d_γ then γ^* has dimension $d_\sigma - d_\gamma$.

Denote by X_{γ^*} the affine toric variety associated to the cone γ^*.

Set $\mu_\gamma^\sigma(e) = \mathrm{ih}^{d_\sigma - d_\gamma - e}(X_{\gamma^*})$, with $e \in \{0, \dots, d_\sigma - d_\gamma\}$. Here $\mathrm{ih}^{d_\sigma - d_\gamma - e}(X_{\gamma^*})$ is the dimension of the $d_\sigma - d_\gamma - e$ intersection cohomology of X_{γ^*}.

The length of M_A^0 is

$$\ell(M_A^0) = \sum_{\gamma \subseteq \sigma} \sum_e \mu_\gamma^\sigma(e) = \sum_{\gamma \subseteq \sigma} \chi(X_\gamma^*).$$

Let P be the polytope given by the intersection of a generic hyperplane with the cone σ. The intersection cohomology of X_{γ^*} can be expressed by face numbers of the polytope P.

f_i number i-dimensional faces of P

$f_{i,j}$ the number of all pairs (i-face,j-face) that are contained in each other.

We get the following lenghts

$$\begin{aligned} d = 3: &\quad \ell(M_A^0) = 3f_0 - 1 \\ d = 4: &\quad \ell(M_A^0) = -2f_0 + 4f_1 \\ d = 5: &\quad \ell(M_A^0) = 7 - 5f_0 - f_2 + 2f_{2,0} \end{aligned}$$

Is there a closed formula for all ranks?

Question 2: How do the simple modules contribute to the holonomic rank of M_A^0?

We have $M_A^0 = \mathrm{FL}(\tilde{h}_+\mathcal{O}_{\tilde{T}})$

$$Gr_{d+e}^W(\tilde{h}_+\mathcal{O}_{\tilde{T}}) = \bigoplus_{\gamma} M^{IC}(X_{\gamma^*}, \mathcal{L}_{\gamma,e}) \quad \text{for } e \in [0, d]$$

What is the holonomic rank of $\mathrm{FL}(M^{IC}(X_{\gamma^*}, \mathcal{L}_{\gamma,e}))$?

More general: Compute holonomic rank of $\mathrm{FL}(M^{IC}(X))$ where X is an affine toric variety.

Question 3: What are the generators of the simple modules in a composition series of M_A^0?

It is known the lowest weight step is a simple module which corresponds to the interior ideal of $\mathbb{C}[\mathbb{N}A]$. One can explicitly describe the generators in all weight steps for dimension 3.

Can one do it in general?

Question 4: Compute the weight filtration on $\mathrm{FL}(M_A^\beta)$ for $\beta \neq 0$.

A. Steiner showed that for $\beta \in \mathbb{Q}^d$ the D-module $\mathrm{FL}(M_A^\beta)$ can be expressed as

$$\mathrm{FL}(M_A^\beta) \simeq \omega_\dagger j_+ \mathcal{O}_{\tilde{T}}^\beta$$

where

$$\tilde{T} \xrightarrow{j} U \xrightarrow{\omega} \mathbb{C}^n$$

and U depends on β. So $\mathrm{FL}(M_A^\beta)$ is a complex mixed Hodge module.

Can one generalize the presented methods to this setting?

Bernstein-Sato polynomials and ν-invariants

ICERM: D-modules, Group Actions, and Frobenius: Computing on Singularities

August 13, 2021

1. Consider the polynomial $f = x^2 + y^2$ over a field $\mathbb{L}$.

   (a) If $\mathbb{L} = \mathbb{F}_p$ where $p > 2$, find $\nu_f^{(x,y)}(p)$, the maximum integer N for which f^N is not in the ideal (x^p, y^p) of $\mathbb{L}[x, y]$.

   (b) If $\mathbb{L} = \mathbb{Q}$, find an element δ of the Weyl algebra

   $$\mathbb{Q}\langle x, y, \partial_x, \partial_y\rangle/(\partial_x x - x\partial_x - 1, \partial_y y - y\partial_y - 1)$$

   and a polynomial b over $\mathbb{Q}$ one variable, such that for all integers s,

   $$\delta \bullet f^{s+1} = b(s)f^s.$$

   (c) Compare $\nu_f^{(x,y)}(p)$ modulo p to the roots of b.

2. Now fix the cusp $f = x^3 - y^2$ over $\mathbb{L}$.

   (a) Suppose that $\mathbb{L} = \mathbb{F}_p$, and $p > 3$. If $p \equiv 1 \bmod 3$, show that $\nu_f(p) = \frac{5p-5}{6}$. Then find a formula for $\nu_f(p)$ when $p \equiv 2 \bmod 3$.

   (b) If $\mathbb{L} = \mathbb{Q}$, consider the polynomial

   $$\delta(z) = -\frac{1}{12}\, x\, \partial_x\, \partial_y^2 + \frac{1}{27}\, \partial_x^3 - \frac{3}{8}\, \partial_y^2 - \frac{1}{4}\, \partial_y^2\, z$$

   over the Weyl algebra, and the polynomial

   $$b(z) = (z + 1)\left(z + \frac{5}{6}\right)\left(z + \frac{7}{6}\right)$$

   over the rational numbers. Verify (if you dare!) that $\delta(s) \bullet f^{s+1} = b(s)f^s$ for all integers s.

   (c) Compare the roots of b to the reductions of your formulas for $\nu_f(p)$ modulo p.

3. Finally, consider $f = x^5 + y^4$ over $\mathbb{L}$.

(a) Suppose that $\mathbb{L} = \mathbb{F}_p$ for some prime p. Find, for some integer m, formulas for $\nu_f^{(x,y)}(p)$ in terms of p when $p \equiv 1 \bmod m$, and when $p \equiv -1 \bmod m$, which are valid for $p \gg 0$.

(b) When $\mathbb{L} = \mathbb{Q}$, the set of roots of the Bernstein-Sato polynomial of f is

$$-\left\{\frac{9}{20}, \frac{13}{20}, \frac{7}{10}, \frac{17}{20}, \frac{9}{10}, \frac{19}{20}, 1, \frac{21}{20}, \frac{11}{20}, \frac{23}{20}, \frac{13}{10}, \frac{27}{20}, \frac{31}{20}\right\}.$$

Which roots do you obtain by taking your formulas for $\nu_f^{(x,y)}(f)$ modulo p?

(c) Try detecting the remaining roots using one, or both, of the following strategies:

- Determine formulas for $\nu_f^{(x,y)}(p)$ for other congruence classes of p modulo m.
- Determine formulas for $\nu_f^{(x^s,y^t)}(p)$, the maximum N for which $f^N \notin (x^{sp}, y^{tp})$, for small values of s and t when $p \equiv \pm 1 \bmod m$.

You may want to use *Macaulay2* or other software as a computational aid!

(d) When $\mathbb{L} = \mathbb{Q}$, the roots of the Bernstein-Sato polynomial of

$$g = x^5 + y^4 + x^3y^2$$

are the same as that for $f = x^4 + y^5$, except the root $-\frac{11}{20}$ is replaced with $-\frac{11}{20} - 1 = -\frac{31}{20}$. Use the techniques you have developed to detect this root.