Friday, August 13, 2021
2:00 – 3:00 EDT
Problem Session

Eloísa Grifo, University of Nebraska -- Lincoln
Claudia Polini, University of Notre Dame
Rebecca R.G., George Mason University
Thomas Reichelt, Universität Heidelberg
Emily Witt, University of Kansas
Lei Wu, KU Leuven
Wenliang Zhang, University of Illinois at Chicago
Question

$k$ field of prime char $p > 0$

$x = (x_{ij})$ generic $n \times m$ matrix

$$R = \frac{k[x]}{I_t(x)}$$

$m = (x_{ij} | 1 \leq i, j \leq n)$

Find an explicit $C > 0$ such that

$$P(Cn) \leq 3^n$$

for all $n \geq 1$, all homogeneous primes $I$

Possible guess: $C = 2$

Why? Because that's the answer in char 0. (!)

If char $k = 0$,

$x = (x_{ij})$ generic $n \times m$ matrix

$$R = \frac{k[x]}{I_t(x)}$$

$m = (x_{ij} | 1 \leq i, j \leq n)$

We know $P(Cn) \leq 3^n$ for all $n \geq 1$, because:
• We saw in Anurag's talk that in this case,

\[ R \cong k[y_2] \xrightarrow{\oplus} k[y, z] \text{ where} \]

- \( y \) is an \( m \times (t-1) \) matrix \( \Rightarrow R \) generated in \( \deg 2 \)
- \( z \) is an \( (t-1) \times n \) matrix

• (Dao-De Stefani-G-Huneke-Núñez Betancourt) (x)

- \( k \) field
- \( R = k[f_1, \ldots, f_e] \xrightarrow{\oplus} S = k[x_1, \ldots, x_d] \)
- \( f_1, \ldots, f_e \) homogeneous, \( \mathcal{D} = \max \deg(f_i) \)
- then

\[ Q^{(\mathcal{D}n)} \leq \eta^n \]

for all homogeneous primes \( Q \) and all \( n \geq 1 \)

**Special Case:**

**Theorem (Carvajal-Rojas—Smolka, 2020)**

- \( \text{char } k = p > 0 \)
- \( R = \frac{k[a, b, c, d]}{(ad - bc)} \xrightarrow{\oplus} k[x_2, x_3] \)
- then \( I^{(2n)} \subseteq I^n \forall n \geq 1, I \) prime of height \( h \)

\[ \Rightarrow I^{(2n)} \subseteq I^n \subseteq \eta^n \Rightarrow c = 2 \text{ works} \]
Cannot do better: \( I = (a, c, d) \) \( b, c = ad \in \mathbb{Z} \Rightarrow C \in \mathbb{Z}^{(2)} \setminus m^2 \)

\[ \Rightarrow C = 1 \] won't work for all primes, so \( C = 2 \) is best possible.

Notice however that in this case, \( R \) is a direct summand of a polynomial ring:

\[ R = \frac{k[a, b, c, d]}{(ad - bc)} \cong k[x, y, u, v] \hookrightarrow k[x, y, u, v] \]

so the fact that we can take \( C = 2 \) also follows from (*)

Follow-up problems:

1) Field of characteristic \( p > 0 \)

2) Symmetric determinantal rings:

\[ k[x]/I_{+}(x) , \quad x \text{ generic } n \times n \text{ symmetric matrix} \]

3) Pfaffian determinantal rings:

\[ k[x]/F_{+}(x) , \quad x \text{ } n \times n \text{ generic alternating matrix} \]
STUDYING SINGULARITIES USING CLOSURE OPERATIONS

REBECCA R.G.

Background:
Recall that for an $R$-module $M$, the trace ideal of $M$ is
\[ \text{tr}_M(R) = \sum_{f \in \text{Hom}_R(M, R)} f(M). \]
When $M$ is finitely-generated, this agrees with the test ideal of the closure operation $\text{cl}_M$.

Problem:
Compute the test/trace ideals of some finitely-generated Cohen-Macaulay modules (also called maximal Cohen-Macaulay modules). The examples given below come from Example 5.25 in Cohen-Macaulay Representations by Leuschke and Wiegand.

1. Let $R = \mathbb{k}[[u^5, u^2v, uv^3, v^5]] \subseteq \mathbb{k}[[u, v]] = S$, where $k$ has characteristic not equal to 5. The indecomposable finitely-generated Cohen-Macaulay $R$-modules are:
   (a) $M_0 = R$,
   (b) $M_1 = R(u^4, uv, v^3) \cong (u^5, u^2v, uv^3),$
   (c) $M_2 = R(u^3, v) \cong (u^5, u^2v),$
   (d) $M_3 = R(u^4, uv^2, v^4) \cong (u^5, u^4v^2, u^3v^4),$ and
   (e) $M_4 = R(u, v^2) \cong (u^5, u^4v^2).$
Compute the trace ideals of these modules in Macaulay2 or by hand. What is the intersection of all of the trace modules?

2. Let $R = \mathbb{k}[[u^8, u^3v, uv^3, v^8]] \subseteq \mathbb{k}[[u, v]] = S$, where $k$ has characteristic not equal to 2. The indecomposable finitely-generated Cohen-Macaulay $R$-modules are:
   (a) $M_0 = R$,
   (b) $M_1 = R(u^7, u^2v, v^3) \cong (u^8, u^2v, uv^3),$  
   (c) $M_2 = R(u^6, uv, v^6) \cong (u^8, u^3v, u^2v^6),$  
   (d) $M_3 = R(u^5, v) \cong (u^8, u^3v),$  
   (e) $M_4 = R(u^4, u^2v^2, v^4) \cong (u^8, u^6v^2, u^4v^4),$  
   (f) $M_5 = R(u^3, uv^2, v^7) \cong (u^8, u^6v^2, u^5v^7),$  
   (g) $M_6 = R(u^2, u^5v, v^2) \cong (u^2v^6, u^5v^7, v^8),$ and
   (h) $M_7 = R(u, v^5) \cong (uv^3, v^8).$
Compute the trace ideals of these modules in Macaulay2 or by hand. What is the intersection of all of the trace modules?

3. Both of these were examples of a more general set of rings. Let $S = \mathbb{k}[[u, v]], r \geq 2$ an integer not divisible by $\text{char}(k)$, and choose $0 < q < r$ with $(q, r) = 1$. Take
$G = \langle g \rangle \cong \mathbb{Z}/r\mathbb{Z}$ to be the cyclic group of order $r$ generated by $g = \begin{pmatrix} \xi_r & 0 \\ 0 & \xi_r^q \end{pmatrix} \in GL(2,k)$, where $\xi_r$ is a primitive $r$th root of unity.

Let $R = k[[u,v]]^G$ be the corresponding ring of invariants, so that $R$ is generated by the monomials $u^a v^b$ satisfying $a+qb \equiv 0 \pmod{r}$. The indecomposable finitely-generated Cohen-Macaulay $R$-modules are

$$M_j = R(u^a v^b \mid a + qb \equiv -j \pmod{r}).$$

(a) Confirm that this gives rise to the first example when $r = 5$ and $q = 3$ and the second example when $r = 8$ and $q = 5$.

(b) For an arbitrary $q$ and $r$, what are the test/trace ideals of the $M_j$? What is the intersection of the trace ideals of all of the $M_j$? You can start with the case $r = q + 1$ (this is done in Benali-Pothagoni-R.G., but via an isomorphism to an $(A_q)$ hypersurface singularity).

(4) A more general open question is: how do the test/trace ideals of finitely-generated Cohen-Macaulay modules correspond to the singularities of the ring? Previous work of the speaker (Perez-R.G.) and of Herzog-Hibi-Stamate shows some cases where there are connections. Choose a class of finitely-generated Cohen-Macaulay modules and compute their trace ideals. Do certain properties of these trace ideals correspond to existing classes of singularities? Or can you find a new class of singularities corresponding to particular behavior of finitely-generated Cohen-Macaulay module trace ideals?

**Relevant Reading:**

Skim through [https://arxiv.org/abs/2103.02529](https://arxiv.org/abs/2103.02529) (Benali-Pothagoni-R.G.). Pay particular attention to Definition 2.10, how they compute examples of test/trace ideals, and how the examples relate to the properties of the rings.


More examples of finitely-generated Cohen-Macaulay modules (including all the ones appearing in Benali-Pothagoni-R.G.) can be found in Cohen-Macaulay Representations by Leuschke and Wiegand, and Cohen-Macaulay Modules over Cohen-Macaulay Rings by Yoshino.

Open problems
Thomas Reichelt

All problems presented deal with GKZ systems $M^0_A$ where $A$ is a $d \times n$ integer matrix which satisfies

1. $ZA = \mathbb{Z}^d$
2. $A$ is saturated, i.e. $\mathbb{R}_{\geq 0} A \cap \mathbb{Z}^d = NA$
3. $A$ is pointed: $NA \cap (-NA) = \{0\}$.

**Question 1**: What is the length of $M^0_A$?

Let $\sigma$ be the cone $\mathbb{R}_{\geq 0} A$, i.e. the cone generated by the columns of $A$. Denote by $\sigma^\vee$ the dual cone and by $\gamma^\perp$ the annihilator of $\gamma$. There is a containment reversing bijection

$$\gamma \leftrightarrow \gamma^* = \gamma^\perp \cap \sigma^\vee$$

If $\gamma$ has dimension $d_\gamma$ then $\gamma^*$ has dimension $d_\sigma - d_\gamma$.

Denote by $X_{\gamma^*}$ the affine toric variety associated to the cone $\gamma^*$. Set $\mu_\gamma^*(e) = \text{ih}^{d_\sigma - d_\gamma - e}(X_{\gamma^*})$, with $e \in \{0, \ldots, d_\sigma - d_\gamma\}$. Here $\text{ih}^{d_\sigma - d_\gamma - e}(X_{\gamma^*})$ is the dimension of the $d_\sigma - d_\gamma - e$ intersection cohomology of $X_{\gamma^*}$.

The length of $M^0_A$ is

$$\ell(M^0_A) = \sum_{\gamma \subseteq \sigma} \sum_e \mu_\gamma^*(e) = \sum_{\gamma \subseteq \sigma} \chi(X_{\gamma^*}).$$

Let $P$ be the polytope given by the intersection of a generic hyperplane with the cone $\sigma$. The intersection cohomology of $X_{\gamma^*}$ can be expressed by face numbers of the polytope $P$.

$f_i$ number $i$-dimensional faces of $P$

$f_{i,j}$ the number of all pairs $(i$-face, $j$-face) that are contained in each other.

We get the following lengths

$$d = 3 : \quad \ell(M^0_A) = 3f_0 - 1$$
$$d = 4 : \quad \ell(M^0_A) = -2f_0 + 4f_1$$
$$d = 5 : \quad \ell(M^0_A) = 7 - 5f_0 - f_2 + 2f_{2,0}$$

Is there a closed formula for all ranks?
Question 2: How do the simple modules contribute to the holonomic rank of $M^0_A$?

We have $M^0_A = \text{FL}(\tilde{h}_+ \mathcal{O}_T)$

$$G^W_{d+e}(\tilde{h}_+ \mathcal{O}_T) = \bigoplus_{\gamma} M^{IC}(X_{\gamma^*}, \mathcal{L}_{\gamma,e}) \quad \text{for} \ e \in [0, d]$$

What is the holonomic rank of $\text{FL}(M^{IC}(X_{\gamma^*}, \mathcal{L}_{\gamma,e}))$?

More general: Compute holonomic rank of $\text{FL}(M^{IC}(X))$ where $X$ is an affine toric variety.

Question 3: What are the generators of the simple modules in a composition series of $M^0_A$?

It is known the lowest weight step is a simple module which corresponds to the interior ideal of $\mathbb{C}[NA]$. One can explicitly describe the generators in all weight steps for dimension 3.

Can one do it in general?

Question 4: Compute the weight filtration on $\text{FL}(M^\beta_A)$ for $\beta \neq 0$.

A. Steiner showed that for $\beta \in \mathbb{Q}^d$ the $D$-module $\text{FL}(M^\beta_A)$ can be expressed as

$$\text{FL}(M^\beta_A) \simeq \omega_\gamma \sigma^\beta \mathcal{O}_T$$

where

$$\tilde{T} \xrightarrow{j} U \xrightarrow{\omega} \mathbb{C}^n$$

and $U$ depends on $\beta$. So $\text{FL}(M^\beta_A)$ is a complex mixed Hodge module.

Can one generalize the presented methods to this setting?
Bernstein-Sato polynomials and $\nu$-invariants

ICERM: $D$-modules, Group Actions, and Frobenius: Computing on Singularities

August 13, 2021

1. Consider the polynomial $f = x^2 + y^2$ over a field $\mathbb{L}$.

(a) If $\mathbb{L} = \mathbb{F}_p$ where $p > 2$, find $\nu_f^{(x,y)}(p)$, the maximum integer $N$ for which $f^N$ is not in the ideal $(x^p, y^p)$ of $\mathbb{L}[x, y]$.

(b) If $\mathbb{L} = \mathbb{Q}$, find an element $\delta$ of the Weyl algebra

$$Q(x, y, \partial_x, \partial_y)/(\partial_x x - x \partial_x - 1, \partial_y y - y \partial_y - 1)$$

and a polynomial $b$ over $\mathbb{Q}$ one variable, such that for all integers $s$,

$$\delta \cdot f^{s+1} = b(s) f^s.$$ 

(c) Compare $\nu_f^{(x,y)}(p)$ modulo $p$ to the roots of $b$.

2. Now fix the cusp $f = x^3 - y^2$ over $\mathbb{L}$.

(a) Suppose that $\mathbb{L} = \mathbb{F}_p$ and $p > 3$. If $p \equiv 1 \mod 3$, show that $\nu_f(p) = \frac{5p-5}{6}$. Then find a formula for $\nu_f(p)$ when $p \equiv 2 \mod 3$.

(b) If $\mathbb{L} = \mathbb{Q}$, consider the polynomial

$$\delta(z) = -\frac{1}{12} x \partial_x \partial_y^2 + \frac{1}{27} \partial_z^3 - \frac{3}{8} \partial_y^2 - \frac{1}{4} \partial_y^2 z$$

over the Weyl algebra, and the polynomial

$$b(z) = (z + 1) \left( z + \frac{5}{6} \right) \left( z + \frac{7}{6} \right)$$

over the rational numbers. Verify (if you dare!) that $\delta(s) \cdot f^{s+1} = b(s) f^s$ for all integers $s$.

(c) Compare the roots of $b$ to the reductions of your formulas for $\nu_f(p)$ modulo $p$. 
3. Finally, consider $f = x^5 + y^4$ over $\mathbb{L}$.

(a) Suppose that $L = \mathbb{F}_p$ for some prime $p$. Find, for some integer $m$, formulas for $\nu_f^{(x,y)}(p)$ in terms of $p$ when $p \equiv 1 \pmod{m}$, and when $p \equiv -1 \pmod{m}$, which are valid for $p \gg 0$.

(b) When $L = \mathbb{Q}$, the set of roots of the Bernstein-Sato polynomial of $f$ is

$$\{ 9, 13, 7, 17, 9, 19, 21, 11, 23, 13, 27, 31 \}.$$  

Which roots do you obtain by taking your formulas for $\nu_f^{(x,y)}(f)$ modulo $p$?

(c) Try detecting the remaining roots using one, or both, of the following strategies:

- Determine formulas for $\nu_f^{(x,y)}(p)$ for other congruence classes of $p$ modulo $m$.
- Determine formulas for $\nu_f^{(x^s,y^t)}(p)$, the maximum $N$ for which $f^N \notin (x^{sp},y^{tp})$, for small values of $s$ and $t$ when $p \equiv \pm 1 \pmod{m}$.

You may want to use Macaulay2 or other software as a computational aid!

(d) When $L = \mathbb{Q}$, the roots of the Bernstein-Sato polynomial of

$$g = x^5 + y^4 + x^3y^2$$

are the same as that for $f = x^4 + y^5$, except the root $-\frac{11}{20}$ is replaced with $-\frac{11}{20} - 1 = -\frac{31}{20}$. Use the techniques you have developed to detect this root.