symbolic powers in mixed characteristic August 2021
(joint work with Alessandro De Stefani and Jack Jeffries)

R A-algebra

the A-linear differential operators on R, $D_{R|A} = \bigcup_{n\geq 0} D^n_{R|A}$, are defined by

1) $D^0_{R|A} = \mathrm{Hom}_R(R,R) \cong R$

2) $D^n_{R|A} = \{ \partial \in \mathrm{Hom}_A(R,R) \mid [\partial, x] = \partial \circ x - x \circ \partial \in D^{n-1}_{R|A} \text{ for all } x \in D^0_{R|A} \}$

I ideal in R

<u>n-th differential power of</u> I

$$I^{\langle n \rangle} := \{ f \in R \mid \partial(f) \in I \text{ for all } \partial \in D^{n-1}_{R|A} \}$$

these have a connection to symbolic powers

the nth <u>symbolic power</u> of $I = \sqrt{I}$ is

$$I^{(n)} := \bigcap_{P \in \mathrm{Ass}(R/I)} (I^n R_P \cap R)$$

throughout: $I = \sqrt{I}$

<u>Lemma</u> R fg A-algebra. Then $I^{(n)} \subseteq I^{\langle n \rangle}$ for all $n \geq 1$

↳ We can sometimes prove theorems about symbolic powers by proving statements about differential powers instead

Theorem (Zariski–Nagata)

$R = k[x_1, \dots, x_d]$, k perfect field

$I = \sqrt{I}$

$$I^{(n)} = I^{<n>} = \bigcap_{\substack{\mathfrak{m} \supseteq I \\ \mathfrak{m} \text{ max}}} \mathfrak{m}^n \qquad \text{for all } n \geq 1$$

$\Rightarrow$ Differential operators are great to study symbolic powers

How about in mixed characteristic?

Example $R = \mathbb{Z}[x]$, $\mathfrak{m} = (2, x)$

$\mathfrak{m}^n = \mathfrak{m}^{(n)}$ for all $n \geq 1$

but

$\partial(2) = 2 \cdot \partial(1) \in (2) \subseteq \mathfrak{m}$ for all $\partial \in D_{R|A}$

$$\Rightarrow \mathfrak{m}^{(n)} \subsetneq \mathfrak{m}^{<n>}$$

From now on: $p \in \mathbb{Z}$ prime

$A = \mathbb{Z}$ or DVR with uniformizer p

R A-algebra

Definition (Joyal, Buium) $p \in \mathbb{Z}$ prime, regular on R

A p-derivation on R is a function $\delta : R \longrightarrow R$ such that:

1) $\delta(1) = 0$
2) $\delta(x+y) = \delta(x) + \delta(y) + \dfrac{x^p + y^p - (x+y)^p}{p}$
3) $\delta(xy) = x^p \delta(y) + \delta(x) y^p + p\, \delta(x) \delta(y)$

Note $\delta(x) = \dfrac{\Phi(x) - x^p}{p}$ is a p-derivation on R $\Leftrightarrow$ $\Phi(x) = x^p + p\,\delta(x)$ is a lift of the Frobenius map on R/p to R

We do have p-derivations when:

- $R = \mathbb{Z}$: $\delta(n) = \dfrac{n - n^p}{p}$ is the unique p-derivation on $\mathbb{Z}$

any p-derivation δ on a $\mathbb{Z}$-algebra satisfies $\delta(n) = \dfrac{n-n^p}{p}$ for $n \in \mathbb{Z}$

- R complete unramified DVR with perfect residue field
- $R = B[x_1, \dots, x_d]$, and B admits a p-derivation

<u>Mixed differential powers</u> (De Stefani – G – Jeffries)

R has a p-derivation δ, $p \in I$:

the nth mixed differential power of I is

$$I^{\langle n \rangle_{mix}} := \{ f \in R \mid \delta^a \circ \partial(f) \in I \text{ for all } a+b < n-1, \partial \in D^b_{R|A} \}$$

<u>theorem</u> (De Stefani – G – Jeffries)

$A = \mathbb{Z}$ or DVR with uniformizer p

R localization of fg smooth A-algebra

R has a p-derivation

Q prime ideal in R, $Q \ni p$, $A/p \hookrightarrow R_Q/QR_Q$ separable

then $Q^{(n)} = Q^{\langle n \rangle_{mix}}$ for all $n \geqslant 1$

Note Over singular rings, we still have $Q^{(n)} \subseteq Q^{\langle n \rangle_{mix}}$

Application Chevalley bounds

Theorem (Chevalley, 1943)
(R, m) complete local ring
$\{I_n\}$ decreasing family of ideals
If $\bigcap_{n \geq 0} I_n = 0$, then $\exists f : \mathbb{N} \to \mathbb{N}$ such that $I_{f(n)} \subseteq m^n$
(so $\{I_n\}$ induces a finer topology than the m-adic topology)

Special Case $I_n = I^{(n)}$, $I = \sqrt{I}$

Uniform Chevalley Lemma (Huneke - Katz - Validashti, 2009)
(R, m) complete local domain
there exists a constant C, independent of I, such that

$$I^{(Cn)} \subseteq m^n$$

Note Finding an explicit C would give a uniform lower bound on the m-adic order of elements in $I^{(n)}$

Theorem (Zariski-Nagata) R regular $\Rightarrow$ can take $C = 1$
so
$$I^{(n)} \subseteq m^n \text{ for all } n \geq 1$$

theorem (Dao – De Stefani – G – Huneke – Núñez Betancourt)

k field

$R = k[f_1, \dots, f_e] \overset{\oplus}{\hookrightarrow} S = k[x_1, \dots, x_d]$ graded direct summand

$m = \underbrace{(f_1, \dots, f_e)}_{\text{homogeneous}}$ $\qquad D := \max_i \{\deg f_i\}$

then $I^{(Dn)} \subseteq m^n$ for all homogeneous $I = \sqrt{I}$ and all $n \geq 1$.

Sketch: $R \underset{\curvearrowleft s}{\hookrightarrow} S$ $\quad s$ graded splitting

1) Show $\eta^{Dn} \cap R \subseteq m^n$ where $\eta = (x_1, \dots, x_d)$

2) Show $I^{(n)} \subseteq \eta^n \cap R$ for all n $\Rightarrow I^{(Dn)} \subseteq \eta^{Dn} \cap R \subseteq m^n$

by

- $\eta^n = \eta^{\langle n \rangle} = \eta^{(n)}$ for all $n \geq 1$
- $f \notin \eta^n \cap R \Rightarrow \partial f \notin \eta$ for some $\partial \in D^{n-1}_{S|k}$

$$\Rightarrow \underbrace{(s \circ \partial)}_{\in D^{n-1}_{R|k} \text{ (Smith)}}(f) \notin m \supseteq I$$

$$\Rightarrow f \notin I^{\langle n \rangle} \supseteq I^{(n)}$$

□

Main obstruction to doing this in mixed characteristic:

to define mixed differential powers, we need a p-derivation!

theorem (De Stefani – G – Jeffries)

A DVR with uniformizer $p \in \mathbb{Z}$ prime

$R = A[f_1, \dots, f_e] \subseteq S = A[x_1, \dots, x_d]$, S has a p-derivation

$\mathfrak{q} = (f_1, \dots, f_e)$ $\qquad$ f_i homogeneous

$\mathfrak{m} = \mathfrak{q} + (p)$ $\qquad$ $D := \max_i \{\deg f_i\}$

① $\left\{ \begin{array}{l} Q \subseteq \mathfrak{q} \text{ prime} \\ R \overset{\oplus}{\hookrightarrow} S \end{array} \right. \implies Q^{(Dn)} \subseteq \mathfrak{q}^n$

② $\left\{ \begin{array}{l} Q \ni p \text{ prime} \\ R/p \overset{\oplus}{\hookrightarrow} S/p \end{array} \right. \implies Q^{(Dn)} \subseteq \mathfrak{m}^n$

therefore, if $R \overset{\oplus}{\hookrightarrow} S$, $I^{(Dn)} \subseteq \mathfrak{m}^n$ for all homogeneous $I = \sqrt{I}$ and $n \geq 1$.

We do have examples showing these can be sharp.

Main tool $\quad$ $R/p \underset{\oplus}{\overset{s}{\rightleftarrows}} S/p$ $\quad$ s splitting , $\bar{x} := x/p$

$Q^{\{n\}} := \{ f \in R \mid s\left(\overline{(\delta^a \circ \partial)(f)}\, S \right) \subseteq \overline{Q} \text{ for all } a+b<n,\ \partial \in D^n_{S|A} \}$