Computing with equivariant $\mathcal{D}$-modules

András Cristian Lőrincz

D-modules, Group Actions, and Frobenius: Computing on Singularities
August 2021, ICERM

Equivariant $\mathcal{D}$-modules

Throughout G is a connected affine algebraic group acting on a smooth complex algebraic variety X.

$$G \times X \underset{pr}{\overset{m}{\rightrightarrows}} X .$$

A $\mathcal{D}_X$-module M is G-equivariant if there is an isomorphism (+ cocycle conditions)

$$m^*(M) \cong pr^*(M), \text{ as } \mathcal{D}_{G\times X}\text{-modules.}$$

$\mathfrak{g}$ – Lie algebra of G, $U\mathfrak{g}$ – universal enveloping algebra.
The action of G on X gives a Lie algebra map $\mathfrak{g} \to \mathcal{D}_X$.

When X is affine, a $\mathcal{D}_X$-module M is equivariant iff the action of $\mathfrak{g}$ on M via $\mathfrak{g} \to \mathcal{D}_X$ can be integrated to a rational G-action.

- $\mathcal{O}_X$ is a G-equivariant $\mathcal{D}_X$-module.
- Functors induced by equivariant maps preserve equivariance.
- If V is a rational G-module, then $\mathcal{P}(V) := \mathcal{D}_X \otimes_{U\mathfrak{g}} V$ is equivariant.

Let $\text{mod}_G(\mathcal{D}_X)$ be the full subcategory of coherent, equivariant $\mathcal{D}$-modules.

The case of finitely many orbits

Suppose G acts on X with finitely many orbits. Then any $M \in \text{mod}_G(\mathcal{D}_X)$ is regular holonomic.

Theorem (Vilonen '94, L.-Walther '19)

The category $\text{mod}_G(\mathcal{D}_X)$ *is equivalent to the category of finite-dimensional representations of a quiver.*

The case of finitely many orbits

Suppose G acts on X with finitely many orbits. Then any $M \in \text{mod}_G(\mathcal{D}_X)$ is regular holonomic.

Theorem (Vilonen '94, L.-Walther '19)

The category $\text{mod}_G(\mathcal{D}_X)$ *is equivalent to the category of finite-dimensional representations of a quiver.*

Let G be reductive and X affine. We want to understand $\text{mod}_G(\mathcal{D}_X)$.
Take $x \in X$ with its orbit Gx closed (such orbit is unique). Then G_x is also reductive, and we have a G_x-module decomposition

$$T_x X \cong T_x(Gx) \oplus V.$$

By Luna's Slice Theorem we have $X \cong G \times_{G_x} V$, so $\text{mod}_G(\mathcal{D}_X) \cong \text{mod}_{G_x}(\mathcal{D}_V)$.

The case of finitely many orbits

Suppose G acts on X with finitely many orbits. Then any $M \in \text{mod}_G(\mathcal{D}_X)$ is regular holonomic.

Theorem (Vilonen '94, L.-Walther '19)

The category $\text{mod}_G(\mathcal{D}_X)$ is equivalent to the category of finite-dimensional representations of a quiver.

Let G be reductive and X affine. We want to understand $\text{mod}_G(\mathcal{D}_X)$.
Take $x \in X$ with its orbit Gx closed (such orbit is unique). Then G_x is also reductive, and we have a G_x-module decomposition

$$T_x X \cong T_x(Gx) \oplus V.$$

By Luna's Slice Theorem we have $X \cong G \times_{G_x} V$, so $\text{mod}_G(\mathcal{D}_X) \cong \text{mod}_{G_x}(\mathcal{D}_V)$.

As V is a semi-simple G_x-module, we first must consider irreducible representations that have finitely many orbits.

Irreducible representations of reductive groups with finitely many orbits have been classified [Sato–Kimura '77, Kac '80].

From now on, we let X be a G-module with G reductive.

Finding the quiver of $\text{mod}_G(\mathcal{D}_X)$

If $S \in \text{mod}_G(\mathcal{D}_X)$ is simple with full support, and f a semi-invariant, then S_f is the *injective* hull of S in $\text{mod}_G(\mathcal{D}_X)$. To understand the $\mathcal{D}$-module structure of S_f, it is essential to compute Bernstein–Sato polynomials.

Finding the quiver of $\text{mod}_G(\mathcal{D}_X)$

If $S \in \text{mod}_G(\mathcal{D}_X)$ is simple with full support, and f a semi-invariant, then S_f is the *injective* hull of S in $\text{mod}_G(\mathcal{D}_X)$. To understand the $\mathcal{D}$-module structure of S_f, it is essential to compute Bernstein–Sato polynomials.

V_λ irreducible G-module with highest weight vector v_λ. $\mathcal{P}(V_\lambda) = \mathcal{D}_X \otimes_{U\mathfrak{g}} V_\lambda$ is a *projective* object in $\text{mod}_G(\mathcal{D}_X)$, as for any $M \in \text{mod}_G(\mathcal{D}_X)$

$$\text{Hom}_{\mathcal{D}_X}(\mathcal{P}(V_\lambda), M) = \text{Hom}_G(V_\lambda, M) = M_\lambda.$$

Thus, character formulas for the simples S in $\text{mod}_G(\mathcal{D}_X)$ are very useful. When X is spherical, then S is multiplicity-free as a G-module [L.-Walther '19].

$\mathcal{P}(V_\lambda) \cong \mathcal{D}_X / (\text{Ann}_{U\mathfrak{g}} v_\lambda)$, which gives an explicit presentation.

Bernstein–Sato polynomial of a polynomial

Theorem (Bernstein '71)

Let $f \in \mathbb{C}[X]$. Then there is an operator $P(s) \in \mathcal{D}_X[s]$ and a nonzero polynomial $b(s) \in \mathbb{C}[s]$ such that

$$P(s) \cdot f^{s+1}(x) = b(s) \cdot f^s(x).$$

Such monic polynomial $b(s)$ of minimal degree is the Bernstein–Sato polynomial $b_f(s)$ of f.

Kashiwara '76: For any polynomial f, the roots of $b_f(s)$ are in $\mathbb{Q}_{<0}$.

Bernstein–Sato polynomial of a polynomial

Theorem (Bernstein '71)

Let $f \in \mathbb{C}[X]$. Then there is an operator $P(s) \in \mathcal{D}_X[s]$ and a nonzero polynomial $b(s) \in \mathbb{C}[s]$ such that

$$P(s) \cdot f^{s+1}(x) = b(s) \cdot f^s(x).$$

Such monic polynomial $b(s)$ of minimal degree is the Bernstein–Sato polynomial $b_f(s)$ of f.

Kashiwara '76: For any polynomial f, the roots of $b_f(s)$ are in $\mathbb{Q}_{<0}$.

Theorem (Sato '60s)

Assume X is prehomogeneous (i.e. with a dense G-orbit). Then for a semi-invariant f

$$f^*(\partial) \cdot f^{s+1}(x) = b_f(s) \cdot f^s(x).$$

Example

Let $f = \det(x_{ij})$ be the $n \times n$ determinant, then

$$\det(\partial_{ij}) \cdot \det(x_{ij})^{s+1} = (s+1) \cdots (s+n) \cdot \det(x_{ij})^s.$$

The case of $m \times n$ matrices [L.-Walther '19]

Take $m \geq n \geq 1$ and let $X = \mathbb{C}^{m \times n}$ be space of $m \times n$ matrices, equipped with the action of $G = \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$.

- $X = \bigcup_{p=0}^{n} O_p$, where O_p is the G-orbit of matrices of rank p .

The case of $m \times n$ matrices [L.-Walther '19]

Take $m \geq n \geq 1$ and let $X = \mathbb{C}^{m \times n}$ be space of $m \times n$ matrices, equipped with the action of $G = \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$.

- $X = \bigcup_{p=0}^{n} O_p$, where O_p is the G-orbit of matrices of rank p .
- We have simples $D_0, D_1, \ldots, D_n$ in $\mathrm{mod}_G(\mathcal{D}_X)$ corresponding to orbits, with $D_n = \mathbb{C}[X]$.

The case of $m \times n$ matrices [L.-Walther '19]

Take $m \geq n \geq 1$ and let $X = \mathbb{C}^{m \times n}$ be space of $m \times n$ matrices, equipped with the action of $G = \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$.

- $X = \bigcup_{p=0}^{n} O_p$, where O_p is the G-orbit of matrices of rank p .
- We have simples $D_0, D_1, \ldots, D_n$ in $\mathrm{mod}_G(\mathcal{D}_X)$ corresponding to orbits, with $D_n = \mathbb{C}[X]$.
- When $m \neq n$, the category is $\mathrm{mod}_G(\mathcal{D}_X)$ is semi-simple.

The case of $m \times n$ matrices [L.-Walther '19]

Take $m \geq n \geq 1$ and let $X = \mathbb{C}^{m \times n}$ be space of $m \times n$ matrices, equipped with the action of $G = \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$.

- $X = \bigcup_{p=0}^{n} O_p$, where O_p is the G-orbit of matrices of rank p .
- We have simples $D_0, D_1, \dots, D_n$ in $\mathrm{mod}_G(\mathcal{D}_X)$ corresponding to orbits, with $D_n = \mathbb{C}[X]$.
- When $m \neq n$, the category is $\mathrm{mod}_G(\mathcal{D}_X)$ is semi-simple.
- When $m = n$, the roots of $b_{\det}(s)$ give a filtration in $\mathrm{mod}_G(\mathcal{D}_X)$:

$$0 \subsetneq \mathbb{C}[X] \subsetneq \mathcal{D}_X \cdot \det^{-1} \subsetneq \cdots \subsetneq \mathcal{D}_X \cdot \det^{-n} = \mathbb{C}[X]_{\det}$$

The simples D_i are given by the successive quotients.
The category $\mathrm{mod}_G(\mathcal{D}_X)$ is given by the quiver

$$\widehat{AA}_n : \quad (0) \rightleftarrows (1) \rightleftarrows \cdots \rightleftarrows (n-1) \rightleftarrows (n)$$

where all the 2-cycles are relations.

The case of $m \times n$ matrices [L.-Walther '19]

Take $m \geq n \geq 1$ and let $X = \mathbb{C}^{m \times n}$ be space of $m \times n$ matrices, equipped with the action of $G = \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$.

- $X = \bigcup_{p=0}^{n} O_p$, where O_p is the G-orbit of matrices of rank p .
- We have simples $D_0, D_1, \dots, D_n$ in $\mathrm{mod}_G(\mathcal{D}_X)$ corresponding to orbits, with $D_n = \mathbb{C}[X]$.
- When $m \neq n$, the category is $\mathrm{mod}_G(\mathcal{D}_X)$ is semi-simple.
- When $m = n$, the roots of $b_{\det}(s)$ give a filtration in $\mathrm{mod}_G(\mathcal{D}_X)$:

$$0 \subsetneq \mathbb{C}[X] \subsetneq \mathcal{D}_X \cdot \det^{-1} \subsetneq \cdots \subsetneq \mathcal{D}_X \cdot \det^{-n} = \mathbb{C}[X]_{\det}$$

The simples D_i are given by the successive quotients.
The category $\mathrm{mod}_G(\mathcal{D}_X)$ is given by the quiver

$$\widehat{AA}_n : \quad (0) \rightleftarrows (1) \rightleftarrows \cdots \rightleftarrows (n-1) \rightleftarrows (n)$$

where all the 2-cycles are relations.

The case of $m \times n$ matrices [L.-Walther '19]

Take $m \geq n \geq 1$ and let $X = \mathbb{C}^{m\times n}$ be space of $m \times n$ matrices, equipped with the action of $G = \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$.

- $X = \bigcup_{p=0}^{n} O_p$, where O_p is the G-orbit of matrices of rank p .
- We have simples $D_0, D_1, \dots, D_n$ in $\mathrm{mod}_G(\mathcal{D}_X)$ corresponding to orbits, with $D_n = \mathbb{C}[X]$.
- When $m \neq n$, the category is $\mathrm{mod}_G(\mathcal{D}_X)$ is semi-simple.
- When $m = n$, the roots of $b_{\det}(s)$ give a filtration in $\mathrm{mod}_G(\mathcal{D}_X)$:

$$0 \subsetneq \mathbb{C}[X] \subsetneq \mathcal{D}_X \cdot \det^{-1} \subsetneq \cdots \subsetneq \mathcal{D}_X \cdot \det^{-n} = \mathbb{C}[X]_{\det}$$

The simples D_i are given by the successive quotients.
The category $\mathrm{mod}_G(\mathcal{D}_X)$ is given by the quiver

$$\widehat{AA}_n : \quad (0) \rightleftarrows (1) \rightleftarrows \cdots \rightleftarrows (n-1) \rightleftarrows (n)$$

where all the 2-cycles are relations.

$\widehat{AA}_n$ has finitely many indecomposable representations!

Local cohomology supported in determinantal varieties [L.-Raicu '20]

$X = \mathbb{C}^{m\times n}$ (with $m \geq n$), with the action of $G = \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$.

When $m \neq n$, the category $\mathrm{mod}_G(\mathcal{D}_X)$ is semi-simple, so each $H^j_{\overline{O}_t}(D_p)$ is a direct sum of $D_0, \dots, D_n$.

In the square case $m = n$, the indecomposables of main interest:

$$Q_p := \frac{\mathbb{C}[X]_{\det}}{\mathcal{D}_X \cdot \det^{p+1-n}} \in \mathrm{mod}_G(\mathcal{D}_X) \text{ corresponds in } \mathrm{rep}(\widehat{AA}_n) \text{ to}$$

$$\mathbb{C} \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{C} \underset{0}{\overset{1}{\rightleftarrows}} \cdots \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{C} \underset{0}{\overset{0}{\rightleftarrows}} 0 \underset{0}{\overset{0}{\rightleftarrows}} \cdots \underset{0}{\overset{0}{\rightleftarrows}} 0 \quad (p \text{ 1's})$$

Direct sum decomposition in square case

q-binomial coefficient: $$\binom{a}{b}_q = \frac{(1-q^a)\cdot(1-q^{a-1})\cdots(1-q^{a-b+1})}{(1-q^b)\cdot(1-q^{b-1})\cdots(1-q)}$$

Theorem (L.-Raicu '20)

We have the direct sum decomposition of $\mathcal{D}$-modules ($t < p$)

$$\sum_{j\geq 0}[H^j_{\overline{O}_t}(D_p)]\cdot q^j = \bigoplus_{s=0}^{t}[Q_s]\cdot q^{(p-t)^2}\cdot m_s(q^2),$$

where $m_t(q) = \binom{n-t}{p-t}_q$, *and for* $s = 0,\cdots,t-1$

$$m_s(q) = \binom{n-s}{p-s}_q \cdot \binom{p-1-s}{t-s}_q - \binom{n-s-1}{p-s-1}_q \cdot \binom{p-2-s}{t-1-s}_q$$

We can compute iterated local cohomology modules, in particular $H^i_{\{0\}}(H^j_{\overline{O}_t}(\mathbb{C}[X]))$ yields the Lyubeznik numbers of $\overline{O}_t$.

We also compute the Borel–Moore homology groups of $\overline{O}_t$ using local cohomology (jt. with Raicu).

Bernstein–Sato polynomials of holonomic functions [L. '21]

An analytic function $h : \Omega \to \mathbb{C}$ on some domain $\Omega \subset X = \mathbb{C}^d$ is holonomic if it is the solution to a holonomic $\mathcal{D}$-module, i.e. the $\mathcal{D}$-module $\mathcal{D}_X \cdot h = \mathcal{D}_X / \operatorname{Ann}_{\mathcal{D}_X} h$ is holonomic.

Examples: $\sin x, \cos x, \exp x, \log x$, algebraic functions, (generalized) hypergeometric functions ${}_pF_q(x)$, Bessel functions, Airy functions, etc.

Bernstein–Sato polynomials of holonomic functions [L. '21]

An analytic function $h : \Omega \to \mathbb{C}$ on some domain $\Omega \subset X = \mathbb{C}^d$ is holonomic if it is the solution to a holonomic $\mathcal{D}$-module, i.e. the $\mathcal{D}$-module $\mathcal{D}_X \cdot h = \mathcal{D}_X / \operatorname{Ann}_{\mathcal{D}_X} h$ is holonomic.

Examples: $\sin x, \cos x, \exp x, \log x$, algebraic functions, (generalized) hypergeometric functions ${}_pF_q(x)$, Bessel functions, Airy functions, etc.

Let $\pi : T^*X \to X$ be the projection. The *singular locus* of a $\mathcal{D}$-module M is

$$\operatorname{Sing} M := \overline{\pi(\operatorname{Char} M \setminus T^*_X X)} \subset X.$$

By Cauchy–Kovalevskaya–Kashiwara, on a simply-connected domain in $X \setminus \operatorname{Sing} M$ the space of analytic solutions of M has dimension $\operatorname{rank} M$.
If M is an $\mathcal{O}_X$-torsion-free holonomic $\mathcal{D}$-module, then $\operatorname{Sing} M$ is a hypersurface.

Bernstein–Sato polynomials of holonomic functions [L. '21]

An analytic function $h : \Omega \to \mathbb{C}$ on some domain $\Omega \subset X = \mathbb{C}^d$ is holonomic if it is the solution to a holonomic $\mathcal{D}$-module, i.e. the $\mathcal{D}$-module $\mathcal{D}_X \cdot h = \mathcal{D}_X / \operatorname{Ann}_{\mathcal{D}_X} h$ is holonomic.

Examples: $\sin x, \cos x, \exp x, \log x$, algebraic functions, (generalized) hypergeometric functions ${}_pF_q(x)$, Bessel functions, Airy functions, etc.

Let $\pi : T^*X \to X$ be the projection. The *singular locus* of a $\mathcal{D}$-module M is

$$\operatorname{Sing} M := \overline{\pi(\operatorname{Char} M \setminus T^*_X X)} \subset X.$$

By Cauchy–Kovalevskaya–Kashiwara, on a simply-connected domain in $X \setminus \operatorname{Sing} M$ the space of analytic solutions of M has dimension rank M.
If M is an $\mathcal{O}_X$-torsion-free holonomic $\mathcal{D}$-module, then $\operatorname{Sing} M$ is a hypersurface.

Definition (L. '21)

Let h be a holonomic function and f the defining polynomial of $\operatorname{Sing} \mathcal{D}_X \cdot h$. Then the monic polynomial $b_h(s) \in \mathbb{C}[s]$ of minimal degree satisfying

$$P(s) \cdot f^{s+1} h = b_h(s) \cdot f^s h,$$

for some $P(s) \in \mathcal{D}_X[s]$, is the Bernstein–Sato polynomial of h.

If h is an integral of an algebraic function, then the roots of $b_h(s)$ are rational.

G-finite functions on prehomogeneous vector spaces

X is a prehomogeneous with dense G-orbit $O = G/H$.

Let $\Gamma = H/H^0$ be the component group, and f the semi-invariant polynomial defining the codimension one component of $X \setminus O$.

There is a 1-to-1 correspondence between irreducible representations χ of Γ and simple equivariant $\mathcal{D}$-modules S^χ with full support.

G-finite functions on prehomogeneous vector spaces

X is a prehomogeneous with dense G-orbit $O = G/H$.

Let $\Gamma = H/H^0$ be the component group, and f the semi-invariant polynomial defining the codimension one component of $X \setminus O$.

There is a 1-to-1 correspondence between irreducible representations χ of Γ and simple equivariant $\mathcal{D}$-modules S^χ with full support.

Fix an arbitrary simply-connected domain $\Omega \subset O$. A function h on Ω is G-finite if $U\mathfrak{g} \cdot h$ is a finite-dimensional G-module. Such h is an algebraic function.

Theorem

The algebra G-finite functions on Ω has a direct sum decomposition into indecomposable $\mathcal{D}_X \times \Gamma$-modules as follows:

$$\bigoplus_{\chi \text{ irrep of } \Gamma} (S^\chi)_f \otimes \chi.$$

Witness representations

Definition

An irreducible G-module V_λ is a *witness representation* for S^χ, if $m_\lambda(S^\chi) \neq 0$ and $\operatorname{rank} \mathcal{P}(\lambda) = \dim \chi$.

V_λ is a witness representation for S^χ iff for any torsion-free equivariant coherent $\mathcal{D}$-module M, we have $[M : S^\chi] = m_\lambda(M)$.

Examples include weights of semi-invariant polynomials.

Witness representations

Definition

An irreducible G-module V_λ is a *witness representation* for S^χ, if $m_\lambda(S^\chi) \neq 0$ and $\operatorname{rank} \mathcal{P}(\lambda) = \dim \chi$.

V_λ is a witness representation for S^χ iff for any torsion-free equivariant coherent $\mathcal{D}$-module M, we have $[M : S^\chi] = m_\lambda(M)$.

Examples include weights of semi-invariant polynomials.

They can be found using Bernstein–Sato polynomials and

$$m_\lambda((S^\chi)_f) = m_\chi(V_\lambda^{*H_0}).$$

This can implicitly give character formulas for $(S^\chi)_f$.

Basic results on Bernstein–Sato polynomials

Let V_λ be a witness representation for S^χ.

Theorem

For non-zero $h \in (S^\chi)_\lambda$ we have

$$f^*(\partial) \cdot f^{s+1} h = b_h(s) \cdot f^s h,$$

with $\deg b_h(s) = \deg f$, and $b_h(s)$ having rational roots.

Basic results on Bernstein–Sato polynomials

Let V_λ be a witness representation for S^χ.

Theorem

For non-zero $h \in (S^\chi)_\lambda$ we have

$$f^*(\partial) \cdot f^{s+1} h = b_h(s) \cdot f^s h,$$

with $\deg b_h(s) = \deg f$, and $b_h(s)$ having rational roots.

Theorem

Assume that $X \setminus O$ is an irreducible hypersurface. Put $d = \deg f$, $n = \dim X$. Then for non-zero $h \in (S^\chi)_\lambda$ and $h^* \in (S^{\chi^*})_{\lambda^*}$, we have

$$b_h(s) \;=\; \pm\, b_{h^*}\left(-s - \frac{n}{d} - 1\right).$$

These generalize Sato's results on semi-invariants.

Classification of G-finite functions

We have to consider only 6 irreducible representations with finitely many orbits (when $Z(f)$ is not normal):

1. $(\mathrm{GL}_2, 3\Lambda_1)$ – binary cubic forms;
2. $(\mathrm{SL}_3 \times \mathrm{GL}_2, 2\Lambda_1 \otimes \Lambda_1)$ – pairs of 3×3 symmetric matrices;
3. $(\mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{GL}_2, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1)$ – $3 \times 3 \times 2$ tensors;
4. $(\mathrm{SL}_6 \times \mathrm{GL}_2, \Lambda_2 \otimes \Lambda_1)$ – pairs of 6×6 skew-symmetric matrices;
5. $(\mathrm{E}_6 \times \mathrm{GL}_2, \Lambda_1 \otimes \Lambda_1)$ – pairs of exceptional simple Jordan algebras;
6. $(\mathrm{SL}_5 \times \mathrm{GL}_4, \Lambda_2 \otimes \Lambda_1)$ – quadruples of 5×5 skew-symmetric matrices.

Binary cubic forms

$G = \mathrm{GL}_2(\mathbb{C})$ acting on $X = \mathrm{Sym}^3 \mathbb{C}^2$, with $\Gamma = S_3$. Let $\mathbb{C}[X] = \mathbb{C}[x_0, x_1, x_2, x_3]$. Let $r = r(x_0, x_1, x_2, x_3)$ be an algebraic function satisfying

$$x_3 \cdot r^3 + x_2 \cdot r^2 + x_1 \cdot r + x_0 = 0.$$

The discriminant is $f = x_1^2 x_2^2 - 4x_1^3 x_3 - 4x_0 x_2^3 - 27x_0^2 x_3^2 + 18x_0 x_1 x_2 x_3$. By M. Sato, the Bernstein–Sato polynomial of the semi-invariant f is

$$b_f(s) = (s+1)^2 \cdot (s+5/6) \cdot (s+7/6), \quad f^*(\partial) \cdot f^{s+1} = b_f(s) \cdot f^s.$$

Proposition

Put $h_0 = x_2 + 3x_3 \cdot r$. Then $h_0 \in S^{\mathrm{st}}$ is a G-finite holonomic function, and $U\mathfrak{g} \cdot h_0 \cong V_{(2,1)}$ is a witness representation for S^{st}. We have

$$f^*(\partial) \cdot f^{s+1} h_0 = b_{h_0}(s) \cdot f^s h_0, \text{ with } \quad b_{h_0}(s) = (s+1)^2 \cdot \left(s + \frac{3}{2}\right)^2$$

The series (2) – (5)

There is a uniform description in terms of Hurwitz algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

For $l = 1, 2, 4, 8$, we have $\dim X = 6l + 6$, and let A denote $\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$, $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$, $\mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$, respectively. Then X is the space of pairs of 3×3 hermitian matrices (X_1, X_2) with the natural action of $\mathrm{SL}_3(A) \times \mathrm{GL}_2(\mathbb{C})$.

The series (2) – (5)

There is a uniform description in terms of Hurwitz algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

For $l = 1, 2, 4, 8$, we have $\dim X = 6l + 6$, and let A denote $\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$, $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$, $\mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$, respectively. Then X is the space of pairs of 3×3 hermitian matrices (X_1, X_2) with the natural action of $\mathrm{SL}_3(A) \times \mathrm{GL}_2(\mathbb{C})$.

The irreducible semi-invariant f of degree 12 is the discriminant of

$$\det(uX_1 + vX_2) = d_1 \cdot u^3 + d_2 \cdot u^2 v + d_3 \cdot uv^2 + d_4 \cdot v^3.$$

We have $\Gamma = S_3$, except the case $l = 1$ when $\Gamma = S_4$.

Theorem

Put $h = h_0(d_1, d_2, d_3, d_4)$. Then $h \in S^{\mathrm{st}}$ is G-finite with $U\mathfrak{g} \cdot h \cong \mathrm{triv} \otimes V_{(2,1)}$ a witness representation for S^{st}. Hence, $f^*(\partial) \cdot f^{s+1} h = b_h(s) \cdot f^s h$ with

$$b_h(s) = (s+1)^2 \left(s + \frac{3}{2}\right)^2 \left(s + \frac{l+1}{2}\right)^2 \left(s + \frac{l+2}{2}\right)^2 \left(s + \frac{3l+8}{12}\right) \cdot \left(s + \frac{3l+10}{12}\right) \left(s + \frac{3l+14}{12}\right) \left(s + \frac{3l+16}{12}\right).$$

d_1, d_2, d_3, d_4 are algebraically independent and $\mathbb{C}[X]^{\mathrm{SL}_3(A)} = \mathbb{C}[d_1, d_2, d_3, d_4]$.

The degenerate case $l = 1$

Extra simple $\mathcal{D}$-modules $S^{(2,1,1)}$ and $S^{(3,1)}$ in the case $l = 1$.

Theorem

The representation $\lambda^1 = \Lambda_1 \otimes (1,1)$ (resp. $\lambda^2 = \Lambda_1 \otimes (-2,-2)$) is a witness representation for $S^{(3,1)}$ (resp. $S^{(2,1,1)}$). For a non-zero $h_1 \in (S^{(3,1)})_{\lambda^1}$ (resp. $h_2 \in (S^{(2,1,1)})_{\lambda^2}$), we have $f^*(\partial) \cdot f^{s+1} h_i = b_{h_i}(s) \cdot f^s h_i$ with

$$b_{h_1}(s) = (s+1)^4 \left(s + \frac{3}{2}\right)^4 \left(s + \frac{5}{6}\right)^2 \left(s + \frac{7}{6}\right)^2,$$

$$b_{h_2}(s) = b_{h_1}\left(s - \frac{1}{2}\right).$$

We can choose $h_1 = \sqrt{f} \cdot h_2$. We use the explicit form of h_1 as follows.

The degenerate case $l = 1$

We think of the space X as pairs of 3×3 symmetric matrices $\{(x_{ij}), (y_{ij})\}$. Write

$$Z = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{22} & x_{23} & x_{33} \\ y_{11} & y_{12} & y_{13} & y_{22} & y_{23} & y_{33} \end{pmatrix}.$$

We number the colums of Z from 0 to 5. Let d_{ij} denote the 2×2 minor of Z formed by the columns i and j, with $0 \leq i \leq j \leq 5$.

The degenerate case $l = 1$

We think of the space X as pairs of 3×3 symmetric matrices $\{(x_{ij}), (y_{ij})\}$. Write

$$Z = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{22} & x_{23} & x_{33} \\ y_{11} & y_{12} & y_{13} & y_{22} & y_{23} & y_{33} \end{pmatrix}.$$

We number the colums of Z from 0 to 5. Let d_{ij} denote the 2×2 minor of Z formed by the columns i and j, with $0 \leq i \leq j \leq 5$.

$$\begin{aligned} v_4 &= d_{04}^2 - d_{03}d_{05} + 2d_{02}d_{23} + 2d_{04}d_{12} + 3d_{12}^2 - 4d_{02}d_{14} + 2d_{01}d_{15}, \\ v_6 &= d_{04}d_{12}^2 + d_{12}^3 + d_{02}d_{05}d_{13} - d_{01}d_{05}d_{14} - 2d_{02}d_{12}d_{14} - d_{02}d_{03}d_{15} + \\ &\quad + d_{01}d_{04}d_{15} + d_{01}d_{12}d_{15} + d_{02}d_{23}d_{12} + d_{02}^2 d_{34}, \\ v_8 &= (-1/3)\cdot(4y_{11}^2 d_2^2 - 12y_{11}^2 d_1 d_3 - 4x_{11}y_{11}d_2 d_3 + 4x_{11}^2 d_3^2 + 36x_{11}y_{11}d_1 d_4 - 12x_{11}^2 d_4 - v_4^2). \end{aligned}$$

Then $y = h_1$ is a root of the algebraic equation

$$y^4 - 2 \cdot v_4 \cdot y^2 + 8 \cdot v_6 \cdot y - v_8 = 0.$$

Series expansions

Based on the work of Gelfand, Kapranov, Zelevinsky on A-hypergeometric systems, the series expansions for roots are given in Sturmfels '00. Below $u = -2v_4, z = -v_8, w = 64v_6^2$:

$$T = u^{-1} - w/u^4 + (2z)/u^3 - (10wz)/u^6 + (6z^2)/u^5 - (70wz^2)/u^8 - (330wz(w^2+7z^3))/u^{12} +$$
$$+(3w^2+20z^3)/u^7 - 12(w^3+35wz^3)/u^{10} + 14(4w^2z+5z^4)/u^9 + 126(5w^2z^2+2z^5)/u^{11} -$$
$$-1716(3w^3z^2+7wz^5)/u^{14} + 11(5w^4+504w^2z^3+84z^6))/u^{13} - 273(w^5+220w^3z^3+220wz^6)/u^{16} +$$
$$+286(7w^4z+147w^2z^4+12z^7)/u^{15} - 1768(7w^5z+330w^3z^4+165wz^7)/u^{18} + 286(140w^4z^2+$$
$$+1008w^2z^5+45z^8)/u^{17} - 25194(12w^5z^2+198w^3z^5+55wz^8)/u^{20} + 4522z^2(14157w^2z^6+$$
$$+156z^9)/u^{23} + 646z(10725w^4z^3+17160w^2z^6+286z^9)/u^{21} + 68(21w^6+8580w^4z^3+$$
$$+27027w^2z^6+715z^9)/u^{19} - 1292(30030w^3z^6+5005wz^9)/u^{22}.$$

Get $b_{h_1}(s)$ from $f^*(\partial)\cdot f^{s+1}h_1 = b_{h_1}(s)\cdot f^s h_1$ by reduction to an invariant subring and then via a careful calculation using the truncation above.