Wednesday August 11, 2021 2:00 – 3:00 EDT

2:00 - 2:07  Daniel Bath, KU Leuven
2:08 - 2:15  Neelima Borade, Princeton University
2:16 - 2:23  Shanna Dobson, University of California at Riverside
2:24 - 2:31  Justin Hilburn, Perimeter Institute for Theoretical Physics
2:32 - 2:39  Mee Seong Im, United States Naval Academy
2:40 - 2:47  Devlin Mallory, University of Michigan
A noncommutative analog of the Peskine–Szpiro Acyclity Lemma

Daniel Bath

Purdue → KU Leuven

August 2021
Peskine–Szpiro Acyclity Lemma

Suppose $R$ is a commutative, Noetherian, local ring and

$$M_\bullet := 0 \to M_q \to M_{q-1} \to \cdots \to M_0$$

a complex of finite $R$-modules such that $\text{depth } M_j \geq j$. If the first nonvanishing homology $H_i$ occurs at $i > 0$, then $\text{depth } H_i \geq 1$.

Key Points

(a) $\text{depth } M = \min\{j \mid \text{Ext}^j_R(R/m, M) \neq 0\}$.

(b) Break up complex into short exact sequences. \textbf{Covariant} $\text{Hom}_R(R/m, -)$ gives progressively smaller lower bounds on depth for kernels and images.

(c) One exact sequence is different & too short:

$$0 \to \ker(M_i \to M_{i-1}) \to M_i \to M_{i-1}.$$  

\textbf{Covariant} $\text{Hom}_R(R/m, -)$ reveals $\text{depth } \ker(M_i \to M_{i-1}) \geq 1$. 

Daniel Bath (Purdue → KU Leuven) A noncommutative analog of the Peskine–Szpiro Acyclity Lemma 

August 2021 2 / 6
Yearnings: Resolutions over non-commutative rings $A$

Commutative Land: rings $R$
- depth zero homology $\implies M_\bullet$ resolves $M_0/(M_1 \to M_0)$.
- Auslander–Buchsbaum: equate depth with projective dimension:
\[
\text{depth } M_i \geq i \iff \text{pdim } M_i \leq \text{depth } R - i.
\]

Noncommutative Land: rings $A$
- Want: criterion on homology $\implies M_\bullet$ resolves $M_0/(M_1 \to M_0)$.
- Without depth, no “suitable” covariant functor ala $\text{Hom}_R(R/m, -)$.
- Projective dimension attached to contravariant functor $\text{Hom}_A(-, A)$.
Definition
The grade $j(M)$ of $M$ is $\min\{j \mid \text{Ext}^j_A(M, A) \neq 0\}$.

Definition
A ring $A$ is an Auslander regular ring provided:
(a) $A$ is Noetherian (both left and right);
(b) $A$ has finite global homological dimension;
(c) $A$ satisfies Auslander’s condition, i.e. for any finitely generated left $A$-module $M$ and for any submodule $N \subseteq \text{Ext}^k_A(M, A)$, the grade of $N$ is bounded below by $j(N) \geq k$.

Proposition (Bjork A.IV Prop 2.2)
Let $A$ be an Auslander regular ring and $M$ a finitely generated left $A$-module. Then
$$j(\text{Ext}^j_A(M, A)) = j(M).$$
Lemma

Let $A$ be an Auslander regular ring and

$$M_\bullet := 0 \to M_m \to M_{m-1} \to \cdots M_1 \to M_0$$

a complex of finite left $A$-modules such that $\text{pdim}(M_q) \leq m - q$. If the first nonvanishing homology occurs at slot $i$, then $j(H_i) \leq m - i < m$.

Conclusions differ: grade $\leftrightarrow$ dimension; depth $\leftrightarrow$ projective dimension.

Sketch

(a) Break up into s.e.s., get progressively smaller upper bounds on pdim.
(b) This stops at different & too short exact sequence

$$0 \to \ker(M_i \to M_{i-1}) \to M_i \to M_i$$

(c) **Contravariant** functor $\text{Hom}_A(-, A)$ less helpful here. Instead: make s.e.s, use Auslander condition & $\text{Ext}_A(\text{Ext}_A(-, A), A)$ kung-fu.
\( \mathcal{A}_R := \mathcal{D}_X \otimes_{\mathbb{C}} R. \) \( \mathcal{D}_X \) algebraic; \( R \) finite, \( \mathbb{C} \)-algebra, regular, domain.

- Think: Bernstein–Sato. \( R = \mathbb{C}[s_1, \ldots, s_r], \mathcal{A}_R = \mathcal{D}_X[s_1, \ldots, s_r]. \)
- Extend order filtration to \( \mathcal{A}_R \) by giving \( R \) weight zero.

**Proposition**

Let \( \text{dim } X = n, \mathcal{D}_X \) be algebraic, and

\[
M_\bullet := 0 \to M_n \to M_{n-1} \to \cdots M_1 \to M_0
\]

a complex of finite left \( \mathcal{A}_R \)-modules such that \( \text{pdim}(M_q) \leq n - q. \) If all the homology modules \( H_i \) for \( i > 0 \) are supported, as \( \mathcal{O}_X \)-modules, on a discrete set, then \( M_\bullet \) resolves \( M_0/(M_1 \to M_0). \)

**Key ideas**

(a) Grade computed on associated graded side. Lemma \( \implies \) lower bound on dim of relative characteristic variety.

(b) Relative characteristic variety is conical in \( \text{gr}(\partial) \) direction. Project to \( X \times \text{Spec } R, \) get upper bound on dim of relative characteristic variety.
Minimal faithful permutation representations of finite groups

Neelima Borade
August 11, 2021

ICERM D-modules, Group Actions, and Frobenius: Computing on Singularities
• Cayley’s theorem guarantees that every finite group is a subgroup of a finite permutation group.

• Degree of $G$ written as $\rho(G)$ is defined to be the smallest natural number $n$ such that $G$ can be embedded in $S_n$. Equivalently, we have a faithful group action of $G$ on $X$, where $|X| = n$.

• Trivial bound is $\rho(G) \leq |G|$ and Cayley’s constant is $\alpha(G) = \frac{\rho(G)}{|G|}$. 
• Johnson’s paper [2] classifies finite groups $G$ such that $p(G) = |G|$, and gives the value of $p(G)$ for Abelian groups.

• Other groups such as $p$-groups, some easy semi-direct products, and some solvable groups, are studied by Elias et. all in [4, 5].

• Galois proved if $q > 11$ is a prime number, then $p(\text{PSL}_2(\mathbb{F}_q)) = q + 1$.

• Table 4 of [1] contains values of $p(G)$ for classical simple groups and exceptional simple groups of Lie type.

• The Atlas [2] contains the value of $p(G)$ for all finite sporadic simple groups.
Simple versus non-simple groups

• For a simple group $G$ computing $p(G)$ is equivalent to finding a subgroup of $G$ of minimal index.
• Given $G$, if $H$ is the subgroup of minimal index in $G$, we obtain a permutation representation of $G$ on $G/H$.
• This permutation representation is, in general, not faithful, unless $G$ is simple.
• In his thesis, Patton [3] determined subgroups of minimal index in $\text{SL}_n(\mathbb{F}_q)$ and $\text{SP}_{2m}(\mathbb{F}_q)$ for $q$ an odd prime power.
• Cooperstein [3] computed the minimal index of a subgroup for the remaining classical groups over finite fields using a generalization of Patton’s method. He listed the size of the kernel of the corresponding permutation representation for each case.
Linear groups

- Write $GL_n$ for the group of $n \times n$ invertible matrices and $SL_n$ for the subgroup of $GL_n$ with determinant 1.
- $\mathbb{F}_q$ denotes the finite field with $q$ elements, where $q$ is taken to be a power of a prime $p$.
- $\{H_1, \ldots, H_n\}$ is called a minimal faithful collection of subgroups if $\text{core}_G(H_1, \ldots, H_n) = \{e\}$ and $\sum_i \frac{|G|}{|H_i|}$ is minimal.
Some results - Takloo-Bighash and B.

\[
\rho(GL_2(\mathbb{F}_q)) = \\
\rho(SL_2(\mathbb{F}_q)) + \sum \text{odd primes } p \text{ (with exponents) s.t. } p \mid q - 1,
\]

\[
\rho(GL_3(\mathbb{F}_q)) = \rho(SL_3(\mathbb{F}_q)) + \\
\sum \text{primes } p \text{ (with exponents) s.t. } p \mid q - 1, p \nmid gcd(3, q - 1),
\]

\[
\rho(GL_n(\mathbb{F}_q)) \leq \rho(SL_n(\mathbb{F}_q)) + \\
\sum \text{primes } p \text{ (with exponents) s.t. } p \mid q - 1, p \nmid gcd(n, q - 1).
\]


Diamonds in Langlands Local Functoriality

Shanna Dobson

University of California, Riverside

ICERM: D-modules, Group Actions, and Frobenius: Computing on Singularities

"Langlands has always viewed the principle of functoriality as central to his view of automorphic representations...local and global Langlands conjectures are special cases of this principle." [Cogdell]

August 5, 2021
Diamond Langlands: Reciprocity Law & Universality [Dob21]

1. Diamond Langlands: $(\infty, 1)$-categorification of Geometric Langlands.

2. Diamond Langlands Functoriality: asserts existence of an $L$-homomorphism $L \mathcal{D}_1^\diamond \to L \mathcal{D}_2^\diamond$ (L-groups of two diamonds over $\mathbb{Q}_p$) that should induce a transfer map from automorphic representations of $\mathcal{D}_1^\diamond$ to $\mathcal{D}_2^\diamond$; reinterpret "p-adic Langlands transfer" as "diamond p-adic Langlands transfer" $Spd_{\mathbb{Q}} = Spa(Q_p^{cycl})/\mathbb{Z}_p^\times$

3. $K^{Efimov}(\mathcal{Y}_{S,E})$ and Diamond $Spd_{\mathbb{Q}_p} = Spa(Q_p^{cycl})/\mathbb{Z}_p^\times$ in $LT/\mathcal{O}_F$

Figure: Diamond $Spd_{\mathbb{Q}_p} = Spa(Q_p^{cycl})/\mathbb{Z}_p^\times$; geometric point $Spa \ C \to \mathcal{D}$
Diamond [Sch17, Dob21]

1. Let $\text{Perf} \subset \text{Perfd}$ be the subcategory of perfectoid spaces of characteristic $p$. A diamond is a pro-étale sheaf $\mathcal{D}$ on site $\text{Perf}$ written as $X/R$ of a perfectoid space $X$ by a pro-étale eqr.

2. Functor of points. Let $C$ be an algebraically closed affinoid field. A geometric point $\text{Spa}(C) \to \mathcal{D}$ is “visible” via pullback along a quasi pro-étale cover $X \to \mathcal{D}$ in profinitely many copies of $\text{Spa}(C)$.
1. Geometrize local Langlands through sheaves on the diamond stack of $G$-bundles on FFC. $E$ is a finite extension of $Q_p$. $G$ is a reductive group over $E$. "Make $\text{Spec}(E)$ geometric."

2. $L^G$ and $\sum$ an algebraic curve, there is an equivalence of derived categories of $D$-modules on the moduli stack of $G$-principal bundles on $\sum$ and quasi-coherent sheaves on the $L^G$-moduli stack of local systems on $\sum$: as $\mathcal{O} \text{Mod} (\text{Loc}_{L^G}(\sum)) \stackrel{\sim}{\rightarrow}$

3. (Fargues) $\text{Bun}_G = \nu$-stack on $\text{Perf}_{\hat{F}_q}$ of $G$-bundles / curve, where $S = \hat{F}_q$, $X_S$ a perfectoid space (adic space / $Q_p$); family of curves.

4. (Hope 2.3) $\text{Bun}_G$ is a smooth diamond stack.

5. At the level of diamonds, $Y_{S,E}^\diamond = S \times (\text{Spa}E)^\diamond$ for $S$ perfectoid.
Diamond Langlands Functoriality in p-adic Groups [Dob21]

1. Local Functoriality: Let $k$ be a local field. If $\phi : L^G \to L^H$ is an $L$-hom, there is a map $\{ (L\text{-packets of}) \text{ admissible reps of } H(k) \} \to \{ (L\text{-packets of}) \text{ admissible reps of } G(k) \}$ $\pi \to \pi'$, given by $c(\pi') = \phi(c(\pi))$ (unram); satake parameters (conjugacy classes); $L$-packet is fiber over an admissible hom $\phi$ (S.M.C).

2. Ludwig: "p-adic Langlands transfer" - p-adic families.

3. Diamond Langlands Functoriality: $L$-hom $L^D_1 \to L^D_2$ induces a transfer map from automorphic reps for $D_1^\diamond$ to $D_2^\diamond$.

4. "p-adic Langlands transfer" as "diamond p-adic Langlands transfer"

$SpdQ_p = Spa(Q_p^{cycl})/\underline{Z_p^\times}$ in $L^D_1 \to \hat{D}_1^\diamond$

5. Universal Construction: $D$-Module/ Functor of Diamond Points $SpaC \to D$. 
Tate’s Thesis and 3d Mirror Symmetry

The global geometric Langlands correspondence is a conjectural equivalence

$$D(Bun_c(C)) \cong \text{Coh}_N(\text{Flat}_G(C))$$

In the case $G = G^L = G_m$ this is a theorem of Lafforgue, Rodnischenk.

There is a less known local Langlands conjecture which claims

$$D(Bun_c(C)) \cong 2\text{Coh}_N(\text{Flat}_{G_L}(C))$$

de Rham sheaves of categories

- There is a stack $X_{dr} = X/x\times x^\wedge x$
  such that

$$D(x) = \text{Coh}(X_{dr})$$

$$2D(x) = 2\text{Coh}(X_{dr})$$

- For $Bun_c(C) = B G(c+1)$ have

$$2D(X) \cong (D(G(c+1)), m^*) \mod$$

In the case $G = G^L = G_m$ Beilinson–Drinfeld proved

$$(D(G_m(c+1)), m^*) \cong (\text{Coh}(\text{Flat}_{G_m}(C)), \otimes)$$

and hence the local Langlands conjecture.
Let's understand Beilinson–Dunkfeld's result:

\[ \text{Flat}_{G}(\mathcal{B}) = \{ (P, \nabla) \mid P \text{ principal } G^\theta\text{-bundle on } \mathcal{B}, \nabla \text{ flat connection}\} \]

\[ = \text{Flat}_{G^\theta}(\mathcal{B}) \]

\[ \text{acting via gauge transformations} \]

\[ g \cdot A = g A g^{-1} + g^{-1} d g \]

For \( G = \mathcal{B}_m \), the result gives:

\[ \text{Flat}_{\mathcal{B}_m}(\mathcal{B}) \cong \text{Flat}_{\mathcal{B}_m}(\mathcal{B}) \times \text{irregular} \]

\[ \cong \mathcal{B}_m \times \mathbb{Z} \times K_1 \times \hat{\mathbb{W}} \]

It is easy to see that:

\[ G \ni (c^+ \rightarrow c^+) \rightarrow \text{kev}(\mathcal{B}_m) \]

Then:

\[ D(\mathbb{Z}) \cong \text{ch}(\mathcal{B}_m) \]

\[ D(K_1) \cong \text{ch}(\text{irregular}) \]

\[ D(\mathcal{B}_m) \cong \text{ch}(A^1/\mathbb{Z}) \]

by Mellin transform.
Kapustin-Witten and Gaitsgory-Frenkel showed that the geometric Langlands correspondences are consequences of S-duality for 4d SYM. Mathematically, this gives rise to a large number of compatibilities that must be satisfied.

For example, work of Gaiotto-Witten shows that

\[ 3d \text{ N=4 theory } Y \text{ with G-flavor symmetry} \]
\[ \rightarrow \]
\[ 3d \text{ N=4 theory } S(Y) \text{ with } G^t \text{-flavor symmetry} \]

Matematically expect

\[ \text{Hamiltonian } G\text{-space} \]
\[ \xymatrix{ G \times X \ar[r] & g^* } \]
\[ X_A \in Z_2 (Bun_G(C\mathbb{D})) \]
\[ X_B \in Z_2 (Fl_{G}(C\mathbb{D})) \]

and for the geometric Langlands equivalence to exchange

\[ X_A \leftrightarrow S(X) B \]
\[ X_B \leftrightarrow S(X) A \]
Based on work of Costello’s students Elliott-Yoo and work of H.-D. Phung, You and I conjectured that when
\[ X = T^* Y \] we have

\[ X_A = D \left( \text{Maps} \left( \overline{\mathcal{D}}, Y \right) \right) \otimes D \left( \text{Maps} \left( \overline{\mathcal{D}}, G \right) \right) \]

\[ X_B = \text{Coh} \left( \text{Maps} \left( \overline{\mathcal{D}}_{dr}, Y/G \right) \right) \otimes \text{Coh} \left( \text{Maps} \left( \overline{\mathcal{D}}_{dr}, BG \right) \right) \]

\[ \{ (p, s) \mid s \in \Gamma (\overline{\mathcal{D}}, Y_p) \text{ with } \overline{\mathcal{D}} s = 0 \} \]

- Horrendous ind-(pre-finite type) stacks
- Singularities are inherently infinite type
- Need to invent new infinite type homological algebra

Weaker conjectures have been made in work of Braverman-Finkelberg-Ginzburg-Troubetzkoy and in forthcoming work of Ben-Zvi-Sakellaridis-Venkatesh. The former is notable for providing proofs in some cases and the latter is notable for providing the number theoretic interpretation.
Sam Raskin and I proved the following when $G = \mathbb{G}^l = \mathbb{G}_m$ and $\mathcal{X} = \mathcal{X}(\mathcal{X}) = \mathbb{T}^r A^1$.

There is an equivalence:

$$\text{DC}(A^1(C^{+})) \cong \text{Con} \left( \text{Maps}(W_{\text{dR}}, A^1/\mathbb{G}_m) \right)$$

intertwining the natural actions of $\text{DC}(\mathbb{G}_m(C^{+})) \cong \text{Con} \left( \text{Maps}(\mathbb{G}_m, C^1) \right)$.

To get a feeling for this result look at

$$\text{Flatt}_{\mathbb{G}_m}(C^{+}) \cong (A^1/k)^{\times} \times \mathbb{G}_m \leftarrow \text{Maps}(W_{\text{dR}}, A^1/\mathbb{G}_m) \cong \mathbb{Z} \times \mathbb{G}_m$$

From a $k$-pro structure on $C(C^{+})$ we see that

$$\text{Maps}(W_{\text{dR}}, A^1/\mathbb{G}_m) \cong \lim_{\to} \begin{pmatrix} \mathcal{L}_n & \mathcal{L}_n \\ \mathcal{L}_n & \mathcal{L}_n \end{pmatrix}^{\infty} \cong \mathbb{Z} \times \mathbb{G}_m$$

Part of our theorem is a fully faithful embedding

$$\text{DC}(A^1) \hookrightarrow \text{Con} \left( \text{Maps}(W_{\text{dR}}, A^1/\mathbb{G}_m) \right)$$

which we believe to be new.
Singularities of a modification of the Grothendieck-Springer resolution
D-modules, Group Actions, and Frobenius: Computing on Singularities
ICERM, Brown University, Providence, RI

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August 8, 2021
Definitions and fundamental constructions, work over $\mathbb{C}$.

$G = \text{GL}_n(\mathbb{C}), \quad g = \text{Lie}(G) = \mathfrak{gl}_n \cong g^*$

$B = $ invertible upper triangular matrices $G, \quad b = \text{Lie}(B)$

$b^* = g/\mathfrak{u}^+, \quad \mathfrak{u}^+ = $ strictly upper triangular matrices in $g$

$G \circlearrowleft G \times b \times \mathbb{C}^n \text{ via } g.(g', r, i) = (g'g^{-1}, r, gi), \quad b.(g', r, i) = (g', brb^{-1}, bi)$,

$G \times B \circlearrowleft G \times b \times \mathbb{C}^n.$

Take the derivative of $G \times B$-action:

$a_{G \times B} : \text{Lie}(G \times B) \to \Gamma(T_{G \times b \times \mathbb{C}^n}) \subseteq \mathbb{C}[T^*(G \times b \times \mathbb{C}^n)],$

$a_{G \times B}(v, w)(g', r, i) = \frac{d}{dt}(\exp(tv), \exp(tw)).(g', r, i)|_{t=0},$

and then dualize $a_{G \times B}$ to obtain $\mu_{G \times B} : T^*(G \times b \times \mathbb{C}^n) \to g^* \times b^*$,

$\quad (g', \theta, r, s, i, j) \mapsto (-\theta + ij, \frac{1}{g'\theta(g')^{-1}} + [r, s]),$

where $\bar{v} : g^* \to b^*$.

$G$-action is free on $\mu_{G \times B}^{-1}(0)$, so we can take $g' = 1$. 
Now, consider $B \circ b \times \mathbb{C}^n$ via $b.(r, i) = (brb^{-1}, bi)$. Take the derivative of $B$-action: $a_B : \text{Lie}(B) \to \Gamma(T_b \times \mathbb{C}^n) \subseteq \mathbb{C}[T^*(b \times \mathbb{C}^n)]$, $a_B(w)(r, i) = \frac{d}{dt} (\exp(tw).(r, i))|_{t=0} = ([w, r], wi)$, and then dualize $a_B$ to obtain $\mu_B : T^*(b \times \mathbb{C}^n) \to b^*, (r, s, i, j) \mapsto [r, s] + ij$. Bijection between $B$-orbits on $\mu_B^{-1}(0)/B \cong \mu_{G \times B}^{-1}(0)/G \times B$ as quotient stacks. From symplectic geometry,

$$
\mu_{G \times B}^{-1}(0)/G \times B \cong T^*((G \times b \times \mathbb{C}^n)/(G \times B)) \cong T^*((G \times_B b \times \mathbb{C}^n)/G)
$$

$$= T^*(\tilde{g} \times \mathbb{C}^n)/G, \text{ where } \tilde{g} \supseteq \tilde{N},$$

$$\tilde{g} = \{(x, b) \in g \times G/B : x \in b\}, \text{ and } \tilde{N} = \{(x, b) \in \mathcal{N} \times G/B : x \in b\}. \text{ So } \mu_B^{-1}(0)/B \cong T^*((\tilde{g} \times \mathbb{C}^n)/G).$$
$\mu_B : T^*(b \times \mathbb{C}^n) \to b^*$ is the Borel moment map of our interest!

Affine quotient $\mu_B^{-1}(0)//B := \text{Spec}(\mathbb{C}[\mu_B^{-1}(0)]^B)$, where

$\mathbb{C}[\mu_B^{-1}(0)] = \frac{\mathbb{C}[T^*(b \times \mathbb{C}^n)]}{\langle \mu_B(r, s, i, j) \rangle}$ and

$\mathbb{C}[\mu_B^{-1}(0)]^B = \{ f \in \mathbb{C}[\mu_B^{-1}(0)] : b.f = f \text{ for all } b \in B \}$.

$\mu_B^{-1}(0)//B$ is a highly singular scheme.

Reminds us of $\mu_G : T^*(g \times \mathbb{C}^n) \to g^*, (r, s, i, j) \mapsto [r, s] + ij$.

GIT quotient: $\mu_G^{-1}(0)//\det G \cong \mu_G^{-1}(0)//\det^{-1}G \cong \text{Hilb}^n(\mathbb{C}^2)$,

Affine quotient: $\mu_G^{-1}(0)//G \cong S^n(\mathbb{C}^2) = (\mathbb{C}^2)^n/S_n$, and

the Hilbert-Chow morphism $\text{Hilb}^n(\mathbb{C}^2) \xrightarrow{\text{HC}} S^n(\mathbb{C}^2)$, which is a symplectic resolution of singularities, i.e., $\text{Hilb}^n(\mathbb{C}^2) = \text{Bl}_I(S^n\mathbb{C}^2)$, where $I$ is an ideal sheaf.
Fundamental constructions.

Let $\chi : B \to \mathbb{C} \setminus \{0\}$ be a nontrivial character.

Work in progress, joint with Meral Tosun (for $n \leq 5$):

1. construct $\mu_B^{-1}(0) \sslash \chi B \to \mu_B^{-1}(0) \sslash B$ explicitly,

2. identify $\mu_B^{-1}(0) \sslash \chi B$ with a noncommutative analog of the Hilbert scheme,

3. identify $\mu_B^{-1}(0) \sslash \chi B$ as a noncommutative blow-up of a closed subscheme of $\mu_B^{-1}(0) \sslash B$, and

4. identify $\mu_B^{-1}(0) \sslash B$ with a noncommutative analog of $S^n(\mathbb{C}^2)$.

For the case when $n \geq 6$ (or the case when we have more than 5 Jordan blocks for a parabolic setting), we have a complete intersection problem!

Thank you! Are there any questions?
Let $R$ be a $k$-algebra and $D_{R} = D_{R/k}$ the ring of $k$-linear differential operators. We’ll study the question of when $R$ is a simple $D_{R}$-module.

**Definition**

A ring $R$ is called $D$-simple if it’s a simple $D_{R}$-module (i.e., given any $r \in R$, there’s $\delta \in D_{R}$ such that $\delta(r) = 1$).

It is easily seen that polynomial rings over $k$, and more generally smooth $k$-algebras, are $D$-simple. Thus, one can think of $D$-simplicity as being a proxy for smoothness.
Examples of $D$-simple rings

Differential operators are hard to compute explicitly, so $D$-simplicity is known only in a few cases over $\mathbb{C}$. Most known examples are direct summands of regular rings (e.g., invariant subrings of finite groups).

Things are better understood in characteristic $p$:

**Theorem (Smith 1995)**

An F-pure ring $R$ is a simple $D_R$-module if and only if $R$ is strongly F-regular.

In case this is meaningful: $F$-regularity is the characteristic-$p$ version of klt singularities.
The previous examples in characteristic 0 and the picture in characteristic-\(p\) suggest that “nicer” singularities might be \(D\)-simple.

For example, one can ask:

**Question (Levasseur and Stafford 1989)**

Let \(R\) be a \(\mathbb{C}\)-algebra. If \(R\) has rational (or klt) Gorenstein singularities, is \(R\) then \(D\)-simple?

Don’t worry about the definitions of these singularities: both are ways of measuring “mildness” of singularities.
A negative answer

Our main theorem is a negative answer to this question:

**Theorem (– 2020)**

*The ring* $R = \mathbb{C}[x, y, z, w]/(x^3 + y^3 + z^3 + w^3)$ *has rational/klt Gorenstein singularities, but is not D-simple.*

Note that $\text{Proj } R$ is a smooth cubic surface; the theorem is true for the coordinate ring of *any* smooth cubic surface.

For the rest of the talk, we’ll discuss where this example comes from, and related examples.
Proof sketch

The idea is fairly simple:

• We look at rings occurring as homogeneous coordinate rings of smooth projective varieties $X$.

• To get rings with “nice” singularities, one has to impose restrictions on the canonical bundle $\omega_X$ of $X$ ($X$ must be Fano).

• We recall a criteria of Hsiao connecting $D$-simplicity to “positivity” (bigness) of the tangent bundle $T_X$.

• Finally, we prove that the tangent bundle of certain Fano surfaces are not big.