Wednesday August 11, 2021 2:00 – 3:00 EDT

2:00	- 2:07	Daniel Bath, KU Leuven
2:08	- 2:15	Neelima Borade, Princeton University
2:16	- 2:23	Shanna Dobson, University of California at Riverside
2:24	- 2:31	Justin Hilburn, Perimeter Institute for Theoretical Physics
2:32	- 2:39	Mee Seong Im, United States Naval Academy
2:40	- 2:47	Devlin Mallory, University of Michigan

A noncommutative analog of the Peskine–Szpiro Acyclity Lemma

Daniel Bath

Purdue $\rightarrow$ KU Leuven

August 2021

Peskine–Szpiro Acyclity Lemma

Suppose R is a commutative, Noetherian, local ring and

$$M_\bullet := 0 \to M_q \to M_{q-1} \to \cdots \to M_0$$

a complex of finite R-modules such that $\operatorname{depth} M_j \geq j$. If the first nonvanishing homology H_i occurs at $i > 0$, then $\operatorname{depth} H_i \geq 1$.

Key Points

(a) $\operatorname{depth} M = \min\{j \mid \operatorname{Ext}^j_R(R/\mathfrak{m}, M) \neq 0\}$.

(b) Break up complex into short exact sequences. Covariant $\operatorname{Hom}_R(R/\mathfrak{m}, -)$ gives progressively smaller lower bounds on depth for kernels and images.

(c) One exact sequence is different & too short:

$$0 \to \ker(M_i \to M_{i-1}) \to M_i \to M_{i-1}.$$

Covariant $\operatorname{Hom}_R(R/\mathfrak{m}, -)$ reveals $\operatorname{depth} \ker(M_i \to M_{i-1}) \geq 1$.

Yearnings: Resolutions over non-commutative rings A

Commutative Land: rings R

- depth zero homology $\implies$ $M_\bullet$ resolves $M_0/(M_1 \to M_0)$.
- Auslander–Buchsbaum: equate depth with projective dimension:

$$\text{depth } M_i \geq i \leftrightsquigarrow \text{pdim } M_i \leq \text{depth } R - i.$$

Noncommutative Land: rings A

- Want: criterion on homology $\implies$ $M_\bullet$ resolves $M_0/(M_1 \to M_0)$.
- Without depth, no "suitable" covariant functor ala $\text{Hom}_R(R/\mathfrak{m}, -)$.
- Projective dimension attached to contravariant functor $\text{Hom}_A(-, A)$.

Definition

The *grade* $j(M)$ of M is $\min\{j \mid \operatorname{Ext}_A^j(M,A) \neq 0\}$.

Definition

A ring A is an *Auslander regular ring* provided:

(a) A is Noetherian (both left and right);

(b) A has finite global homological dimension;

(c) A satisfies *Auslander's condition*, i.e. for any finitely generated left A-module M and for any submodule $N \subseteq \operatorname{Ext}_A^k(M,A)$, the grade of N is bounded below by $j(N) \geq k$.

Proposition (Bjork A.IV Prop 2.2)

Let A be an Auslander regular ring and M a finitely generated left A-module. Then

$$j(\operatorname{Ext}_A^{j(M)}(M,A)) = j(M).$$

Lemma

Let A be an Auslander regular ring and

$$M_\bullet := 0 \to M_m \to M_{m-1} \to \cdots M_1 \to M_0$$

a complex of finite left A-modules such that $\mathrm{pdim}(M_q) \leq m - q$. If the first nonvanishing homology occurs at slot i, then $j(H_i) \leq m - i < m$.

Conclusions differ: grade $\leftrightsquigarrow$ dimension; depth $\leftrightsquigarrow$ projective dimension.

Sketch

(a) Break up into s.e.s., get progressively smaller upper bounds on pdim.

(b) This stops at different & too short exact sequence

$$0 \to \ker(M_i \to M_{i-1}) \to M_i \to M_i$$

(c) Contravariant functor $\mathrm{Hom}_A(-, A)$ less helpful here. Instead: make s.e.s, use Auslander condition & $\mathrm{Ext}_A(\mathrm{Ext}_A(-, A), A)$ kung-fu.

- $\mathscr{A}_R := \mathscr{D}_X \otimes_{\mathbb{C}} R$. $\mathscr{D}_X$ algebraic; R finite, $\mathbb{C}$-algebra, regular, domain.
- Think: Bernstein–Sato. $R = \mathbb{C}[s_1, \ldots, s_r]$, $\mathscr{A}_R = \mathscr{D}_X[s_1, \ldots, s_r]$.
- Extend order filtration to $\mathscr{A}_R$ by giving R weight zero.

Proposition

Let $\dim X = n$, $\mathscr{D}_X$ be algebraic, and

$$M_\bullet := 0 \to M_n \to M_{n-1} \to \cdots M_1 \to M_0$$

a complex of finite left $\mathscr{A}_R$-modules such that $\operatorname{pdim}(M_q) \leq n - q$. If all the homology modules H_i for $i > 0$ are supported, as $\mathscr{O}_X$-modules, on a discrete set, then $M_\bullet$ resolves $M_0/(M_1 \to M_0)$.

Key ideas

(a) Grade computed on associated graded side. Lemma $\implies$ lower bound on dim of relative characteristic variety.

(b) Relative characteristic variety is conical in $\operatorname{gr}(\partial)$ direction. Project to $X \times \operatorname{Spec} R$, get upper bound on dim of relative characteristic variety.

Minimal faithful permutation representations of finite groups

Neelima Borade

August 11, 2021

ICERM D-modules, Group Actions, and Frobenius: Computing on Singularities

Algebraic setup

- Cayley's theorem guarantees that every finite group is a subgroup of a finite permutation group.
- Degree of G written as $\rho(G)$ is defined to be the smallest natural number n such that G can be embedded in S_n. Equivalently, we have a faithful group action of G on X, where $|X| = n$.
- Trivial bound is $\rho(G) \leq |G|$ and Cayley's constant is $\alpha(G) = \frac{\rho(G)}{|G|}$.

History

- Johnson's paper [2] classifies finite groups G such that $p(G) = |G|$, and gives the value of $p(G)$ for Abelian groups.
- Other groups such as p-groups, some easy semi-direct products, and some solvable groups, are studied by Elias et. all in [4, 5].
- Galois proved if $q > 11$ is a prime number, then $p(\mathrm{PSL}_2(\mathbb{F}_q)) = q + 1$.
- Table 4 of [1] contains values of $p(G)$ for classical simple groups and exceptional simple groups of Lie type.
- The Atlas [2] contains the value of $p(G)$ for all finite sporadic simple groups.

Simple versus non-simple groups

- For a simple group G computing $p(G)$ is equivalent to finding a subgroup of G of minimal index.
- Given G, if H is the subgroup of minimal index in G, we obtain a permutation representation of G on G/H.
- This permutation representation is, in general, *not* faithful, unless G is simple.
- In his thesis, Patton [3] determined subgroups of minimal index in $\mathrm{SL}_n(\mathbb{F}_q)$ and $\mathrm{SP}_{2m}(\mathbb{F}_q)$ for q an odd prime power.
- Cooperstein [3] computed the minimal index of a subgroup for the remaining classical groups over finite fields using a generalization of Patton's method. He listed the size of the kernel of the corresponding permutation representation for each case.

Linear groups

- Write GL_n for the group of *nxn* invertible matrices and SL_n for the subgroup of GL_n with determinant 1.
- $\mathbb{F}_q$ denotes the finite field with q elements, where q is taken to be a power of a prime p.
- $\{H_1, \ldots, H_n\}$ is called a minimal faithful collection of subgroups if $\mathrm{core}_G(H_1, \ldots, H_n) = \{e\}$ and $\Sigma_i \frac{|G|}{|H_i|}$ is minimal.

Some results - Takloo-Bighash and B.

$$\rho(GL_2(\mathbb{F}_q)) =$$
$$\rho(SL_2(\mathbb{F}_q)) + \Sigma \text{ odd primes p (with exponents) s.t. } p \mid q-1,$$
$$\rho(\mathrm{GL}_3(\mathbb{F}_q)) = \rho(\mathrm{SL}_3(\mathbb{F}_q)) +$$
$$\Sigma \text{ primes p (with exponents) s.t. } p \mid q-1, p \nmid gcd(3, q-1),$$
$$\rho(GL_n(\mathbb{F}_q)) \leq \rho(SL_n(\mathbb{F}_q)) +$$
$$\Sigma \text{ primes p (with exponents) s.t. } p \mid q-1, p \nmid gcd(n, q-1).$$

References

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References

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Diamonds in Langlands Local Functoriality

Shanna Dobson

University of California, Riverside

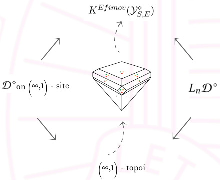
ICERM: D-modules, Group Actions, and Frobenius: Computing on Singularities

"Langlands has always viewed the principle of functoriality as central to his view of automorphic representations...local and global Langlands conjectures are special cases of this principle." [Cogdell]

August 5, 2021

Diamond Langlands: Reciprocity Law & Universality [Dob21]

1. Diamond Langlands: $(\infty, 1)$-categorification of Geometric Langlands.
2. Diamond Langlands Functoriality: asserts existence of an L-homomorphism ${}^L\mathcal{D}_1^\diamond \to {}^L\mathcal{D}_2^\diamond$ (L-groups of two diamonds over $\mathbb{Q}_p$) that should induce a transfer map from automorphic representations of $\mathcal{D}_1^\diamond$ to $\mathcal{D}_2^\diamond$; reinterpret "p-adic Langlands transfer" as "diamond p-adic Langlands transfer" $SpdQ_p = Spa(Q_p^{cycl})/\underline{Z_p^\times}$



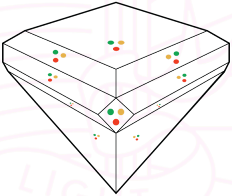
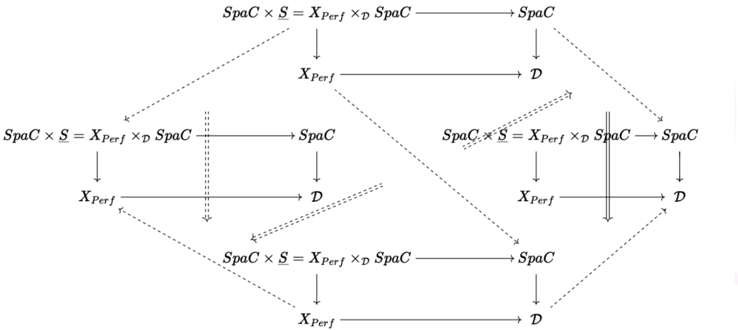


Figure: Diamond $SpdQ_p = Spa(Q_p^{cycl})/\underline{Z_p^\times}$; geometric point $Spa\ C \to \mathcal{D}$

3. $K^{Efimov}(\mathcal{Y}^\diamond_{S,E})$ and Diamond $SpdQ_p = Spa(Q_p^{cycl})/\underline{Z_p^\times}$ in $LT/\mathcal{O}_F$

Diamond [Sch17, Dob21]

1. Let $Perf \subset Perfd$ be the subcategory of perfectoid spaces of characteristic p. A diamond is a pro-'etale sheaf $\mathcal{D}$ on site $Perf$ written as X/R of a perfectoid space X by a pro-étale eqr.
2. Functor of points. Let C be an algebraically closed affinoid field. A geometric point $Spa(C) \to \mathcal{D}$ is "visible" via pullback along a quasi pro-étale cover $X \to \mathcal{D}$ in profinitely many copies of $Spa(C)$.



Diamonds in Geometrization of Local Langlands [SchFar] + D-Modules in Geometric Langlands

1. Geometrize local Langlands through sheaves on the diamond stack of G-bundles on FFC. E is a finite extension of Q_p. G is a reductive group over E. "Make $Spec(E)$ geometric."
2. ^LG and $\sum$ an algebraic curve, there is an equivalence of derived categories of D-modules on the moduli stack of G-principal bundles on $\sum$ and quasi-coherent sheaves on the ^LG-moduli stack of local systems on $\sum$: as $\mathcal{O}Mod(Loc_{^LG}(\sum) \xrightarrow{\simeq}$
3. (Fargues) $Bun_G = v$-stack on $Perf_{\hat{F}_q}$ of G-bundles / curve, where $S = \hat{F}_q$, X_S a perfectoid space (adic space / Q_p); family of curves.
4. (Hope 2.3) Bun_G is a smooth diamond stack.
5. At the level of diamonds, $Y^{\diamond}_{S,E} = S$ x $(SpaE)^{\diamond}$ for S perfectoid.

Diamond Langlands Functoriality in p-adic Groups [Dob21]

1. Local Functoriality: Let k be a local field. If $\phi : {}^L G \to {}^L H$ is an L-hom, there is a map $\{$ (L-packets of) admissible reps of H(k)$\} \to \{$ (L-packets of) admissible reps of $G(k)\}$ $\pi \to \pi^{'}$, given by $c(\pi^{'}) = \phi(c(\pi))$ (unram); satake parameters (conjugacy classes); L-packet is fiber over an admissible hom ϕ (S.M.C) .
2. Ludwig: "p-adic Langlands transfer" - p-adic families.
3. Diamond Langlands Functoriality: L -hom ${}^L\mathcal{D}_1^\diamond \to {}^L\mathcal{D}_2^\diamond$ induces a transfer map from automorphic reps for $\mathcal{D}_1^\diamond$ to $\mathcal{D}_2^\diamond$.
4. "p-adic Langlands transfer" as "diamond p-adic Langlands transfer" $SpdQ_p = Spa(Q_p^{cycl})/\underline{Z_p^\times}$ in ${}^L\mathcal{D}_1^\diamond$ $\hat{\mathcal{D}}_1^\diamond$
5. Universal Construction: D-Module/ Functor of Diamond Points $SpaC \to \mathcal{D}$.

Tate's Thesis and 3d Mirror Symmetry

The global geometric Langlands correspondence is a conjectural equivalence

$$D(\mathrm{Bun}_G(C)) \cong \mathrm{Coh}_{\mathcal{N}}(\mathrm{Flat}_{G^L}(C))$$

In the case $G = G^L = \mathbb{G}_m$ this is a theorem of Laumon, Rothstein.

There is a less known local Langlands conjecture which claims

$$2D(\mathrm{Bun}_G(\mathring{\mathbb{D}})) \cong 2\mathrm{Coh}_{\mathcal{N}}(\mathrm{Flat}_{G^L}(\mathring{\mathbb{D}}))$$

de Rham sheaves of categories

- There is a stack $X_{dR} = X / X \times \hat{X}_{\Delta X}$ such that

$$D(X) = \mathrm{Coh}(X_{dR})$$
$$2D(X) = 2\mathrm{Coh}(X_{dR})$$

- For $\mathrm{Bun}_G(\mathring{\mathbb{D}}) = BG((t))$ have

$$2D(X) \cong (D(G((t))), m_*)\text{-mod}$$

coherent sheaves of categories

- $2\mathrm{Coh}(\mathrm{Spec}\, R) = R$-linear categories

$$= (R\text{-mod}, \otimes)\text{-mod}$$

- More generally a stack is called 1-affine if

$$2\mathrm{Coh}(X) \cong (\mathrm{Coh}(X), \otimes)\text{-mod}$$

- $\mathrm{Flat}_{G^L}(\mathring{\mathbb{D}})$ is 1-affine

In the case $G = G^L = \mathbb{G}_m$ Beilinson–Drinfeld proved

$$(D(\mathbb{G}_m((t))), m_*) \cong (\mathrm{Coh}(\mathrm{Flat}_{\mathbb{G}_m}(\mathring{\mathbb{D}})), \otimes)$$

and hence the local Langlands conjecture.

Let's understand Beilinson–Drinfeld's result

$$\mathrm{Flat}_{G^L}(\mathring{\mathbb{D}}) = \{ (P, \nabla) \mid P \text{ principal } G^L\text{-bundle on } \mathring{\mathbb{D}},\ \nabla \text{ flat connection} \}$$

must be trivial $\qquad \nabla = d - A$

$$= \mathfrak{g}^L((t))\,dt \,/\, G^L((t))$$

$G^L((t))$ acting via gauge transformations
$g \cdot A = gAg^{-1} + g^{-1}dg$

For $G = \mathbb{G}_m$ Levelt–Turritin gives

$$\mathrm{Flat}_{\mathbb{G}_m}(\mathring{\mathbb{D}}) \cong \mathrm{Flat}_{\mathbb{G}_m}(\mathring{\mathbb{D}})_{R.S.} \times \text{irregular}$$

$$\mathrm{Flat}_{\mathbb{G}_m}(\mathring{\mathbb{D}})_{R.S.} = \mathbb{C}\tfrac{dt}{t} / \mathbb{Z}\tfrac{dt}{t} \times B\mathbb{G}_m, \qquad \text{irregular} = \ker\left(\mathrm{Res}: \mathbb{C}((t))dt / \mathbb{C}[[t]]dt \to \mathbb{C}\right)_{dR}$$

it is also easy to see that

$$\mathbb{G}_m((t)) \cong \mathbb{G}_m \times \mathbb{Z} \times K_1 \times \hat{W}$$

$\mathbb{Z} = \pi_1(\mathbb{G}_m)$

K_1: first congruence subgroup $\ker(\mathbb{G}_m[[t]] \to \mathbb{G}_m)$

$\hat{W}$: formal group of big witt vectors, $D(\hat{W}) = 0$

Then

$$D(\mathbb{Z}) \cong \mathrm{Coh}(B\mathbb{G}_m)$$

$$D(K_1) \cong \mathrm{Coh}(\text{irregular}) \quad \text{since } t\mathbb{C}[[t]] \text{ is dual to } t^{-2}\mathbb{C}[t^{-1}]$$

$$D(\mathbb{G}_m) \cong \mathrm{Coh}(\mathbb{A}^1/\mathbb{Z}) \quad \text{by Mellin Transform}$$

Kapustin-Witten and Gaitsgory-Frenkel showed that the geometric Langlands correspondences are consequences of S-duality for 4d SYM. Mathematically, this gives rise to a large number of compatibilities that must be satisfied.

For example work of Gaiotto-Witten shows that

$$
\begin{array}{ccc}
\text{3d } N=4 \text{ theory } \mathcal{T} \text{ with} & \xleftrightarrow{\text{S-dual}} & \text{3d } N=4 \text{ theory } S(\mathcal{T}) \text{ with} \\
G\text{-flavor symmetry} & & G^L\text{-flavor symmetry} \\
\cup\!| & & \cup\!| \\
\text{Hamiltonian } G\text{-space} & & \text{Hamiltonian } G^L\text{-space}
\end{array}
$$

generally not preserved by S-duality

Mathematically expect

$$
\begin{array}{ccc}
\text{Hamiltonian } G\text{-space} & & X_A \in 2D(\mathrm{Bun}_G(\mathring{D})) \\
& \longrightarrow & \\
G \circlearrowright X \xrightarrow{\mu} \mathfrak{g}^* & & X_B \in 2\mathrm{Coh}_{\mathcal{N}}(\mathrm{Flat}_G(\mathring{D}))
\end{array}
$$

and for the geometric Langlands equivalence to exchange

$$X_A \longleftrightarrow S(X)_B$$

$$X_B \longleftrightarrow S(X)_A$$

Based on work of Costello's students Elliott–Yoo and work of H.–Dimofte, Philsang Yoo and I conjectured that when $X = T^*Y$ we have

$$X_A = D(\underbrace{\mathrm{Maps}(\mathring{\mathbb{D}}, Y)}_{Y((t))}) \circlearrowleft D(\underbrace{\mathrm{Maps}(\mathring{\mathbb{D}}, G)}_{G((t))})$$

$$X_B = \mathrm{Coh}(\underbrace{\mathrm{Maps}(\mathring{\mathbb{D}}_{dR}, Y/G)}_{\{(P, \nabla, s) \mid s \in \Gamma(\mathring{\mathbb{D}}, Y_P) \text{ with } \nabla s = 0\}}) \circlearrowleft \mathrm{Coh}(\underbrace{\mathrm{Maps}(\mathring{\mathbb{D}}_{dR}, BG)}_{\mathrm{Flat}_G(\mathring{\mathbb{D}})})$$

- Horrendous ind-(pw-finite type) stack
- singularities are inherently infinite type
- Need to invent new infinite type homological algebra

weaker conjectures have been made in work of Braverman–Finkelberg–Ginzburg–Travkin and in forthcoming work of Ben-Zvi–Sakellaridis–Venkatesh. The former is notable for providing proofs in some cases and the latter is notable for providing the number theoretic interpretation.

Sam Raskin and I proved the following when $G = G^L = \mathbb{G}_m$ and $X = \delta(X) \cong T^* \mathbb{A}^1$.

<u>Thm</u> (Tate's thesis in de Rham context / Abelian 3d mirror symmetry)
There is an equivalence

$$D(\mathbb{A}^1((t))) \cong \mathrm{Coh}(\mathrm{Maps}(\mathring{\mathbb{D}}_{dR}, \mathbb{A}^1/\mathbb{G}_m))$$

intertwining the natural actions of $D(\mathbb{G}_m((t))) \cong \mathrm{Coh}(\mathrm{Flat}_{\mathbb{G}_m}(\mathring{\mathbb{D}}))$.

To get a feeling for this result look at

$$\begin{tikzcd}
\mathrm{Flat}_{\mathbb{G}_m}(\mathring{\mathbb{D}})_{R.S.} = (\mathbb{A}^1/\mathbb{Z})^{\frac{dt}{t}} \times B\mathbb{G}_m & & \mathrm{Maps}(\mathring{\mathbb{D}}_{dR}, \mathbb{A}^1/\mathbb{G}_m)_{R.S.} \arrow[ll] \\
\{d - \lambda \frac{dt}{t}\} \arrow[u] & & \{s \in \mathbb{C}((t)) \mid (d - \lambda \frac{dt}{t}) s = 0\} \arrow[ll] \arrow[u]
\end{tikzcd}$$

$$\{s \in \mathbb{C}((t)) \mid (d - \lambda \tfrac{dt}{t}) s = 0\} = \begin{cases} 0, & \lambda \notin \mathbb{Z} \\ \mathbb{C}t^{\lambda}, & \lambda \in \mathbb{Z} \end{cases}$$

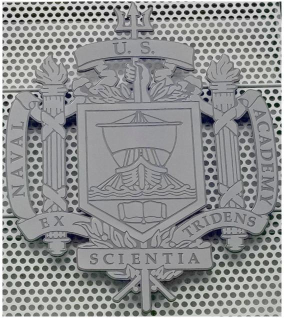
From ind-pro structure on $\mathbb{C}((t))$ we see that

$$\mathrm{Maps}(\mathring{\mathbb{D}}_{dR}, \mathbb{A}^1/\mathbb{G}_m)_{R.S.} = \Big[\operatorname*{colim}_n \lim_m \big(\cdots \lambda \big)\Big] \Big/ \mathbb{Z} \times \mathbb{G}_m$$

Part of our theorem is a fully faithful embedding

$$D(\mathbb{A}^1) \hookrightarrow \mathrm{Coh}(\mathrm{Maps}(\mathring{\mathbb{D}}_{dR}, \mathbb{A}^1/\mathbb{G}_m)_{R.S.})$$

which we believe to be new.



Singularities of a modification of the Grothendieck-Springer resolution

D-modules, Group Actions, and Frobenius:
Computing on Singularities
ICERM, Brown University, Providence, RI

Mee Seong Im

United States Naval Academy

August 8, 2021

Definitions and fundamental constructions, work over $\mathbb{C}$.

$G = \mathrm{GL}_n(\mathbb{C})$, $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{gl}_n \cong \mathfrak{g}^*$
$B =$ invertible upper triangular matrices G, $\mathfrak{b} = \mathrm{Lie}(B)$
$\mathfrak{b}^* = \mathfrak{g}/\mathfrak{u}^+$, $\mathfrak{u}^+ =$ strictly upper triangular matrices in $\mathfrak{g}$
$G \circlearrowright G \times \mathfrak{b} \times \mathbb{C}^n$ via $g.(g', r, i) = (g'g^{-1}, r, gi)$,
$B \circlearrowright G \times \mathfrak{b} \times \mathbb{C}^n$ via $b.(g', r, i) = (g', brb^{-1}, bi)$,
$G \times B \circlearrowright G \times \mathfrak{b} \times \mathbb{C}^n$.
Take the derivative of $G \times B$-action:

$a_{G\times B} : \underbrace{\mathrm{Lie}(G \times B)}_{\mathfrak{g}\times\mathfrak{b}} \to \Gamma(T_{G\times\mathfrak{b}\times\mathbb{C}^n}) \subseteq \mathbb{C}[T^*(G \times \mathfrak{b} \times \mathbb{C}^n)]$,

$a_{G\times B}(v, w)(g', r, i) = \frac{d}{dt}(\exp(tv), \exp(tw)).(g', r, i)|_{t=0}$,
and then dualize $a_{G\times B}$ to obtain $\mu_{G\times B} : T^*(G \times \mathfrak{b} \times \mathbb{C}^n) \to \mathfrak{g}^* \times \mathfrak{b}^*$,

$$(g', \theta, r, s, i, j) \mapsto (-\theta + ij,\ \overline{g'\theta(g')^{-1}} + [r, s]),$$

where $\overline{v} : \mathfrak{g}^* \twoheadrightarrow \mathfrak{b}^*$.
G-action is free on $\mu^{-1}_{G\times B}(0)$, so we can take $g' = 1$.

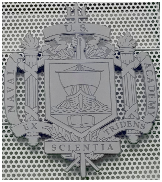
Fundamental constructions.

Now, consider $B \circlearrowright \mathfrak{b} \times \mathbb{C}^n$ via $b.(r,i) = (brb^{-1}, bi)$.
Take the derivative of B-action: a_B : Lie$(B) \to \Gamma(T_{\mathfrak{b}\times\mathbb{C}^n}) \subseteq \mathbb{C}[T^*(\mathfrak{b}\times\mathbb{C}^n)]$,
$a_B(w)(r,i) = \frac{d}{dt}(\exp(tw).(r,i))|_{t=0} = ([w,r], wi)$, and then dualize a_B to obtain $\mu_B : \underbrace{T^*(\mathfrak{b}\times\mathbb{C}^n)}_{\mathfrak{b}\times\mathfrak{b}^*\times\mathbb{C}^n\times(\mathbb{C}^n)^*} \to \mathfrak{b}^*$, $(r,s,i,j) \mapsto [r,s] + ij$.

$\mu_B^{-1}(0) \hookrightarrow \mu_{G\times B}^{-1}(0)$, $(r,s,i,j) \mapsto (1, ij, r, s, i, j)$.
Bijection between B-orbits on $\mu_B^{-1}(0)$ and $G\times B$-orbits on $\mu_{G\times B}^{-1}(0)$.
So $\mu_B^{-1}(0)/B \cong \mu_{G\times B}^{-1}(0)/G\times B$ as quotient stacks.
From symplectic geometry,

$$\mu_{G\times B}^{-1}(0)/G\times B \cong T^*((G\times\mathfrak{b}\times\mathbb{C}^n)/(G\times B)) \cong T^*((G\times_B \mathfrak{b}\times\mathbb{C}^n)/G)$$
$$= T^*((\widetilde{\mathfrak{g}}\times\mathbb{C}^n)/G), \text{ where } \widetilde{\mathfrak{g}} \supseteq \widetilde{\mathcal{N}},$$

$\widetilde{\mathfrak{g}} = \{(x,\mathfrak{b}) \in \mathfrak{g}\times G/B : x \in \mathfrak{b}\}$, and $\widetilde{\mathcal{N}} = \{(x,\mathfrak{b}) \in \mathcal{N}\times G/B : x\in\mathfrak{b}\}$.
So $\mu_B^{-1}(0)/B \cong T^*((\widetilde{\mathfrak{g}}\times\mathbb{C}^n)/G)$.



Fundamental constructions.

$\mu_B : T^*(\mathfrak{b} \times \mathbb{C}^n) \to \mathfrak{b}^*$ is the Borel moment map of our interest!
Affine quotient $\mu_B^{-1}(0) /\!\!/ B := \operatorname{Spec}(\mathbb{C}[\mu_B^{-1}(0)]^B)$, where
$\mathbb{C}[\mu_B^{-1}(0)] = \dfrac{\mathbb{C}[T^*(\mathfrak{b} \times \mathbb{C}^{\mathfrak{n}})]}{\langle \mu_B(r,s,i,j) \rangle}$ and

$$\mathbb{C}[\mu_B^{-1}(0)]^B = \{ f \in \mathbb{C}[\mu_B^{-1}(0)] : b.f = f \text{ for all } b \in B \}.$$

$\mu_B^{-1}(0) /\!\!/ B$ is a highly singular scheme.
Reminds us of $\mu_G : T^*(\mathfrak{g} \times \mathbb{C}^n) \to \mathfrak{g}^*, (r,s,i,j) \mapsto [r,s] + ij$.
GIT quotient: $\mu_G^{-1}(0) /\!\!/_{\det} G \cong \mu_G^{-1}(0) /\!\!/_{\det^{-1}} G \cong \mathrm{Hilb}^n(\mathbb{C}^2)$,
Affine quotient: $\mu_G^{-1}(0) /\!\!/ G \cong S^n(\mathbb{C}^2) = (\mathbb{C}^2)^n / S_n$, and
the Hilbert-Chow morphism $\mathrm{Hilb}^n(\mathbb{C}^2) \overset{\mathrm{HC}}{\twoheadrightarrow} S^n(\mathbb{C}^2)$, which is a symplectic resolution of singularities, *i.e.*, $\mathrm{Hilb}^n(\mathbb{C}^2) = \mathrm{Bl}_{\mathscr{I}}(S^n\mathbb{C}^2)$, where $\mathscr{I}$ is an ideal sheaf.

Fundamental constructions.

Let $\chi : B \to \mathbb{C} \setminus \{0\}$ be a nontrivial character.

Work in progress, joint with Meral Tosun (for $n \leq 5$):

1. construct $\mu_B^{-1}(0) /\!\!/_\chi B \twoheadrightarrow \mu_B^{-1}(0) /\!\!/ B$ explicitly,
2. identify $\mu_B^{-1}(0) /\!\!/_\chi B$ with a noncommutative analog of the Hilbert scheme,
3. identify $\mu_B^{-1}(0) /\!\!/_\chi B$ as a noncommutative blow-up of a closed subscheme of $\mu_B^{-1}(0) /\!\!/ B$, and
4. identify $\mu_B^{-1}(0) /\!\!/ B$ with a noncommutative analog of $S^n(\mathbb{C}^2)$.

For the case when $n \geq 6$ (or the case when we have more than 5 Jordan blocks for a parabolic setting), we have a complete intersection problem!

Thank you! Are there any questions?

D-simplicity of singular rings

Let R be a k-algebra and $D_R = D_{R/k}$ the ring of k-linear differential operators. We'll study the question of when R is a simple D_R-module.

Definition

A ring R is called D-simple if it's a simple D_R-module (i.e., given any $r \in R$, there's $\delta \in D_R$ such that $\delta(r) = 1$).

It is easily seen that polynomial rings over k, and more generally smooth k-algebras, are D-simple.

Thus, one can think of D-simplicity as being a proxy for smoothness.

Examples of D-simple rings

Differential operators are hard to compute explicitly, so D-simplicity is known only in a few cases over $\mathbb{C}$.

Most known examples are direct summands of regular rings (e.g., invariant subrings of finite groups).

Things are better understood in characteristic p:

Theorem (Smith 1995)

An F-pure ring R is a simple D_R-module if and only if R is strongly F-regular.

In case this is meaningful: F-regularity is the characteristic-p version of klt singularities.

D-simplicity as measure of singularity

The previous examples in characteristic 0 and the picture in characteristic-p suggest that "nicer" singularities might be D-simple.

For example, one can ask:

Question (Levasseur and Stafford 1989)

Let R be a $\mathbb{C}$-algebra. If R has rational (or klt) Gorenstein singularities, is R then D-simple?

Don't worry about the definitions of these singularities: both are ways of measuring "mildness" of singularities.

A negative answer

Our main theorem is a negative answer to this question:

Theorem (– 2020)

The ring $R = \mathbb{C}[x, y, z, w]/(x^3 + y^3 + z^3 + w^3)$ has rational/klt Gorenstein singularities, but is not D-simple.

Note that $\operatorname{Proj} R$ is a smooth cubic surface; the theorem is true for the coordinate ring of *any* smooth cubic surface.

For the rest of the talk, we'll discuss where this example comes from, and related examples.

Proof sketch

The idea is fairly simple:

- We look at rings occurring as homogeneous coordinate rings of smooth projective varieties X.
- To get rings with "nice" singularities, one has to impose restrictions on the canonical bundle ω_X of X (X must be Fano).
- We recall a criteria of Hsiao connecting D-simplicity to "positivity" (bigness) of the tangent bundle T_X.
- Finally, we prove that the tangent bundle of certain Fano surfaces are not big.