Thursday, August 12, 2021
2:00 – 3:00 EDT

2:00 - 2:07	Swaraj Pande, University of Michigan
2:08 - 2:15	Afshan Sadiq, University of Sussex
2:16 - 2:23	Jyoti Singh, Vnit, Nagpur
2:24 - 2:31	Sudeshna Roy, Chennai Mathematical Institute

Multiplicities of Jumping Numbers

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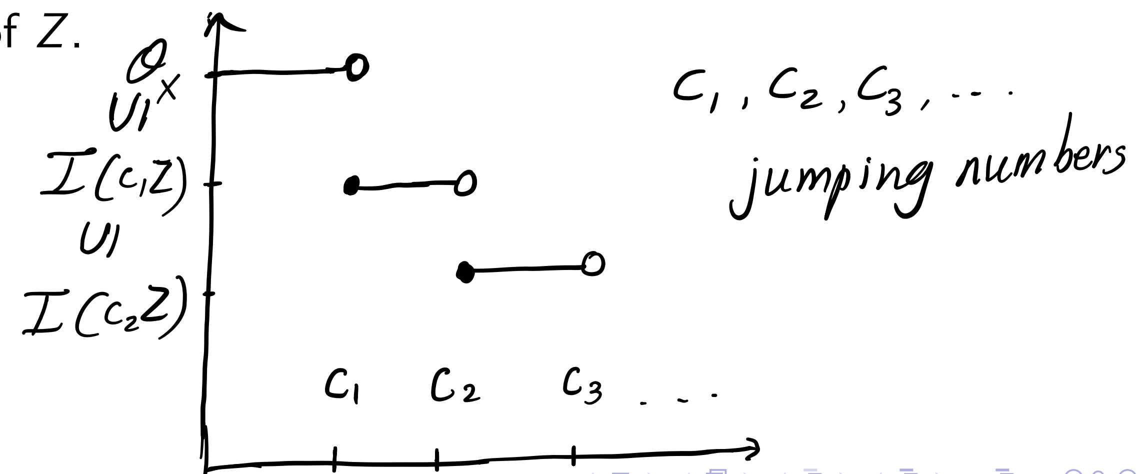
D-modules, Group Actions, and Frobenius: Computing on Singularities, ICERM
August 12, 2021

Multiplier Ideals

Let X be a smooth variety over $\mathbb{C}$, Z a subscheme of X. Then,

- There is a decreasing family of ideals $\mathcal{I}(c \cdot Z)$ called *multiplier ideals of Z* parametrized by $c \in \mathbb{R}_{>0}$.
- $\mathcal{I}(c \cdot Z)$ quantifies the singularities of Z in X.
- *Jumping numbers* of Z in X are a discrete set of rational numbers where the family $\mathcal{I}(c \cdot Z)$ changes. These are interesting invariants of the singularity of Z.
-

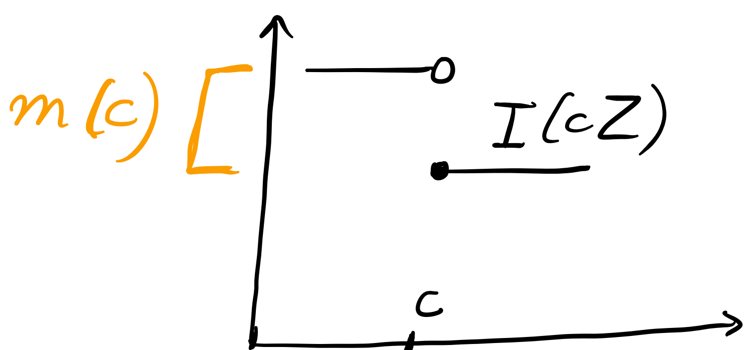


Multiplicity of Jumping numbers

Suppose Z is a point scheme at a (closed) point $x \in X$. And let c be a jumping number of Z. Then,

- $\mathcal{I}(c \cdot Z)$ is co-supported at x.
- $\mathcal{I}(c \cdot Z) \subsetneq \mathcal{I}((c - \varepsilon) \cdot Z)$ since c is a jumping number.
- Hence, we can "measure the jump at c" and define the *multiplicity of* c to be:

$$m(c) := \lambda(\mathcal{I}(c - \varepsilon \cdot Z)_x / \mathcal{I}(c \cdot Z)_x)$$

- 

Main Theorem

Theorem 1

Let Z be a closed point subscheme in X and c be any jumping number of Z. Then the sequence of multiplicities $(m(c+n))_{n\in\mathbb{N}}$ is polynomial in n of degree less than the dimension X.

- This theorem is a finiteness property for the family $\mathcal{I}(c \cdot Z)$, similar to Skoda's theorem.
- This generalizes the work when X is a surface of Alberich-Carramiñana, Àlvarez Montaner, Dachs-Cadefau and González-Alonso.

Highest possible degree

Question

Let c be a jumping number of Z. Then what is the degree of $m(c+n)$? In particular, when is it the highest possible ($= \dim X - 1$)?

Theorem 2

The degree of $m(c+n)$ is $d-1$ if and only if $c+d-1$ is a jumping number contributed by a *Rees valuation* of Z (where $d = \dim X$).

- This theorem says that the valuations needed to compute such jumping numbers are the most basic valuations associated to Z (from the divisors that appear on the normalized blow-up of Z).
- When Z is defined by a monomial ideal, all jumping numbers are of this form.

Thank you!

Thank you for your time!

Link to the arxiv preprint: `https://arxiv.org/abs/2102.07080`

Primary Decomposition of Binomial Modules

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August 11, 2021

History

- Primary decomposition of ideals plays a significant role in commutative algebra and is closely linked to algebraic geometry because it gives a decomposition of a variety into irreducible varieties.
- In case of modules it represents a module as intersection of primary modules whose annihilators have as radical a prime ideal.
- Lasker Noether decomposition theorem ensures that every module can be represented as intersection of a finite number of primary modules.
- In the present study we will discuss the techniques and methods of primary decomposition of submodules of free modules over a polynomial ring in n variables.

- There are three main algorithms to compute primary decomposition of ideals in polynomial rings over the rational numbers.
- The first algorithm was given by Gianni, Trager, Zacharias. In this algorithm the primary decomposition is done by reducing the problem to the zero-dimensional case. The ideas of Gianni, Trager, Zacharias were generalized by Rutman ([7]) to primary decomposition of submodules of a free module.
- Eisenbud, Huneke and Vasconcelos use homological methods to reduce the problem of the primary decomposition to the equidimensional case.

  They characterize the intersection of primary modules of a module M of maximal dimension as the kernel of the canonical map $M \to Ext_R^c(Ext_R^c(M, R), R)$, c the codimension of M. The primary decomposition is described for ideals.

- The method of primary decomposition knowing the isolated prime ideals was given by Shimoyama, Yokoyama ([8]). In this method they introduced pseudo–primary ideals and extract the primary components from these ideals. An ideal is pseudo–primary if its radical is a prime ideal. This method was generalized for submodules of a free module by Idrees ([4]).
- With these methods we obtain a primary decomposition with possible redundant primary ideals.
- The method to compute the primary decomposition without having redundant primary ideals was given by Noro and Kawazoe([6]). They introduced saturated separating ideals, ideals obtained as the quotient of an ideal with its isolated primary component. This method is more effluent for some examples.
- The aim of this paper is to generalize the idea of Noro and Kawazoe to primary decomposition of modules.

Introduction

Let K be a field and $R = K[x_1, \ldots, x_n]$, let $e_i = \begin{pmatrix} \mathbf{0} \\ 1 \\ \mathbf{0} \end{pmatrix}$-$i$ be the $i-th$ unit vector in R^m. A monomial in R^m is an expression $x^\alpha e_i$, $x^\alpha := x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$. A binomila is an expression of the form $ax^\alpha e_i + bx^\beta e_j$, $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$, $a, b \in K$, $1 \leq i, j \leq m$. A binomial module is a submodule of R^m generated by binomials. The aim of this paper is to prove that a binomial module has a primary decomposition into binomial primary modules and the associated primes are binomial ideals. The idea is to generalize the paper of Eisenbud and Strumfels ([2])

Lemma

Let $<$ be a monomial ordering on R suitably extended to R^m and $M \subseteq R^m$ a binomial module. Then the reduced Gröbner basis consist of binomials and the normal form of a term is again a term.

Corollary

Let M be a binomial module and $m_1, \ldots, m_k$ monomials in R^m, $f_1, \ldots, f_l \in R$ such that $\sum f_i m_i \in M$. Let f_{ij} be the terms of f_i. Then either $f_{ij} m_i \in M$ or there exist $f_{i'j'} \neq f_{ij}$ and $a \in K$ such that $f_{ij} m_i + a f_{i'j'} m'_i \in M$

Corollary

Let M be a binomial module and m a monomial then $M : m$ is binomial.

Definition

A module $M \subseteq R^m$ is called cellular, if $M : R^m$ is a cellular ideal, i.e, every variable is a non-zero divisor mod $M : R^m$ or nilpotent. M is called a lattice module, if $M : (x_1, \ldots, x_n) = M$.

Remark

If M is a binomial than the cellular decomposition consist also of binomial modules. Therefore it is enough to prove that cellular binomial module have a binomial primary decomposition.

Lemma

Every binomial module $M \subseteq S^m$ *is isomorphic to* $S^l \bigoplus I_1 \bigoplus \ldots \bigoplus I_k$, $I_v \subseteq S$ *binomial ideals.*

Corollary

Assume K *is algebraically closed. The prime ideals associated to* M *are binomial.*

Corollary

M *has a primary decomposition of binomial modules.*

Corollary

A binomial lattice module $M \subseteq R * m$ *has a binomial primary decomposition and the associated primes are binomial.*

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Eisenbud, D.; Huneke, C.; Vasconcelos, W.: Direct Methods for Primary Decomposition. Inventiones Mathematicae 110, 207–235 (1992).

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Gianni, P.; Trager, B.; Zacharias, G.: Gröbner Bases and Primary Decomposition of Polynomial Ideals. Journal of Symbolic Computation 6, 149–167 (1988).

Idrees, N.: Algorithms for primary decoposition of modules. Studia Scientiarum Mathematicarum Hungarica 48 (2), 227-246 (2011).

Gerhard Pfister, G.; Sadiq, A.; Steidel, S.: An Algorithm for Primary Decomposition in Polynomial Rings over the Integers. Central European Journal of Mathematics Vol. 9, No. 4, (2010) 897-904

Kawazoe, T., Noro, M., Algorithms for computing a primary ideal decomposition without producing intermediate redundant components, Journal of Symbolic Computation 46 (2011) 11581172

E.W. Rutman: Gröbner bases and primary decomposition of modules. J. Symbolic Computation (1992)14, 483-503.

The End

Derived fuctors of graded local cohomology modules

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(joint work with Prof. Tony J. Puthenpurakal)

D-modules, Group Actions, and Frobenius: Computing on Singularities
ICERM

August 09-13, 2021

Motivation

- Let $R = K[x_1, \ldots, x_n]$. It is well known that $H^i_m(H^j_I(R))$, for an ideal I of R, is isomorphic to a direct sum of a finite number of copies of E (the ungraded injective hull of R/m).
- Huneke-Sharp, Trans. Amer. Math. Soc. 1993: In the case $char(K) > 0$
- Lyubeznik, Invent. Math. 1993: In the case $char(K) = 0$.
- Ma- Zhang, Math. Res. Lett. 2014: Every graded F-module is Eulerian. $\tau(R)$ is Eulerian. $H^i_m(\tau(R)) \cong \oplus E(n)^c$.

Question

What happens when $charK = 0$?

Weyl Algebra and D-module

- The n^{th} *Weyl algebra*, denoted by $A_n(K)$(or A_n), is defined as the K-subalgebra $K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$ of $End_K(K[X])$ generated by the K-linear endomorphisms $x_1, \ldots, x_n$ and $\partial_1, \ldots, \partial_n$ of $K[X]$ given by $x_i(f) = x_i f$ and $\partial_i(f) = \frac{\partial f}{\partial x_i}$ for all $f \in K[X]$.
- R_f is D-module.
- Check complex $0 \to R \to \oplus R_{f_i} \to \ldots \to R_{f_1,\ldots,f_k} \to 0$ is a complexes of D-modules.
- If M is a D-module, so is $C(\underline{\mathrm{f}}, M)$
- If M is a D-module, so is local cohomology modules $H^i_{\underline{\mathrm{f}}}(M)$.
- A finitely generated D-module M is said to be holonomic if either $M = 0$ or $d(M) = n$

Generalized Eulerian $A_n(K)$-modules

$$\mathcal{E}_n := X_1\partial_1 + \ldots + X_n\partial_n.$$

Definition

Let M be a graded $A_n(K)$-module and z be a homogeneous element of M. Then M is said to be *Eulerian* if $\mathcal{E}_n z = |z| \cdot z$.

Definition

A graded $A_n(K)$-module M is said to be *generalized Eulerian* if for a homogeneous element z of M there exists a positive integer a such that $(\mathcal{E}_n - |z|)^a \cdot z = 0$.

Property 1 (Puthenpurakal)

Let $0 \to M_1 \xrightarrow{\alpha_1} M_2 \xrightarrow{\alpha_2} M_3 \to 0$ be a short exact sequence of graded $A_n(K)$-modules. Then M_2 is generalized Eulerian if and only if M_1 and M_3 are generalized Eulerian.

Theorem [Puthenpurakal, ___]

Let M be a nonzero generalized Eulerian $A_n(K)$-module. Then $S^{-1}M$ is also a generalized Eulerian $A_n(K)$-module for each homogeneous system $S \subseteq R$.

Corollary [Puthenpurakal, ___]

Let $I = (f_1, \ldots, f_s)$ be a homogeneous ideal in R with f_i's are homogeneous polynomials in R. Let M be a nonzero generalized Eulerian $A_n(K)$-module. Then $H^i_I(M)$ is a generalized Eulerian $A_n(K)$-module.

Theorem [Puthenpurakal, ___]

Let K be a field of characteristic zero. Let $R = K[X_1, \ldots, X_n]$ be standard graded. Let $\mathcal{T}$ be a graded Lyubeznik functor on $^*Mod(R)$. Then $\mathcal{T}(R)$ is a generalized Eulerian $A_n(K)$-module.

Corollary [Puthenpurakal,___]

Let K be a field of characteristic zero. Let $R = K[X_1, \ldots, X_n]$ be standard graded. Let $\mathcal{T}$ be a graded Lyubeznik functor on ${}^*Mod(R)$. Then $H^i_{\mathfrak{m}}\mathcal{T}(R) = E(n)^{a_i}$ for some $a_i \geq 0$.

Theorem

Fix $\nu \geqslant 0$. Then the graded K-vector space $Tor_\nu^{A_n(K)}(M^\sharp, N)$ is concentrated in degree $-n$, i.e., $Tor_\nu^{A_n(K)}(M^\sharp, N)_j = 0$ for all $j \neq -n$.

Conjecture

Fix $\nu \geqslant 0$. The graded K-vector space $Ext^\nu_{A_n(K)}(M, N)$ is concentrated in degree zero, i.e., $Ext^\nu_{A_n(K)}(M, N)_j = 0$ for $j \neq 0$.

THANK YOU

On derived functors of graded local cohomology modules - II

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Joint work with Tony J. Puthenpurakal (IIT Bombay), and Jyoti Singh (VNIT Nagpur)

D-modules, Group Actions, and Frobenius: Computing on Singularities

August 9, 2021

Notations and definitions

K: a field of characteristic zero.

$R := K[X_1, \ldots, X_n]$ is a polynomial ring over K.

${}^*E_R(K)$: the graded injective hull of K over R.

$(-)^\vee = {}^*\operatorname{Hom}_K(-, K)$: the graded Matlis dual.

$\mathbf{C}(\underline{f}, R)$: the Čech complex on R w.r.t. $\underline{f} := f_1, \ldots, f_r$.

$H^i_I(R) = H^i(\mathbf{C}(\underline{f}, R))$: the i^{th} local cohomology module of R w.r.t. I.

$A_n(K) = R\langle \partial_1, \ldots, \partial_n \rangle$: n^{th}-Weyl algebra over K, where $\partial_i = \partial / \partial X_i$.

$\mathbf{K}^\bullet\big(\underline{\partial}; H^i_I(R)\big)$: the de Rham complex, where $\underline{\partial} := \partial_1, \ldots, \partial_n$.

$H^\nu(\underline{\partial}; W) = H^\nu(\mathbf{K}^\bullet(\underline{\partial}; W))$: the ν^{th} de Rham cohomology module.

M: a left $A_n(K)$-module.

$M^\sharp$: the standard right $A_n(K)$-module associated to M.

- A f.g. left $A_n(K)$-module is holonomic if it is zero or its Bernstein dimension is n.

Motivation

Consider standard gradings on both R and $A_n(K)$.
Fix $i, j \geq 0$. Let $I, J \subseteq R$ be homogeneous ideals.

➢ (Lyubeznik, 1993). $H^i_I(R)$ are holonomic.

➢ (Björk). $H^\nu\big(\underline{\partial}; H^i_I(R)\big)$ are f.d. graded K-vector spaces $\forall\ \nu \geq 0$.

➢ (Puthenpurakal, 2015). For all $\nu \geq 0$,

$$H^\nu\big(\underline{\partial}; H^i_I(R)\big)_l = 0 \quad \text{for } l \neq -n.$$

⋆ Notice $H^\nu(\underline{\partial}; H^i_I(R)) \cong \operatorname{Tor}^{A_n(K)}_{n-\nu}\big(H^0_{\{0\}}(R)^\sharp, H^i_I(R)\big)$.

➢ (Puthenpurakal-Singh, 2018).For all $\nu \geq 0$,

$$\operatorname{Tor}^{A_n(K)}_\nu\big(H^j_J(R)^\sharp, H^i_I(R)\big)_l = 0 \quad \text{for } l \neq -n.$$

➢ (Björk). For all $\nu \geq 0$, $\operatorname{Tor}^{A_n(K)}_\nu\big(H^j_J(R)^\sharp, H^i_I(R)\big)$ and $\operatorname{Ext}^\nu_{A_n(K)}\big(H^j_J(R), H^i_I(R)\big)$ are f.d. graded K-vector spaces.

Main result

Theorem (Puthenpurakal, - , and Singh, 2020).
For all $\nu \geq 0$*; the graded* K*-vector space* $\operatorname{Ext}^{\nu}_{A_n(K)}\left(H^j_J(R), H^i_I(R)\right)$ *is concentrated in degree* 0*, i.e.,*

$$\operatorname{Ext}^{\nu}_{A_n(K)}\left(H^j_J(R), H^i_I(R)\right)_l = 0 \quad \text{for } l \neq 0.$$

Question. *Let* M, N *be holonomic* $A_n(K)$*-modules. Is* ${}^*\operatorname{Ext}^{\nu}_R(M, N)$ *holonomic?*

⋆ Need not be true.

➢ (Switala-Zhang, 2018). Let $n \geq 2$. Take $M = H^1_{(X_1)}(R)$. Then M is holonomic, but ${}^*\operatorname{Hom}_R(M(-n), {}^*E_R(K))$ is not holonomic.

Application

Corollary. If M and N are graded holonomic generalized Eulerian left $A_n(K)$-modules, then any extension

$$\eta_i : \ 0 \to N(i) \to Y \to M \to 0$$

splits for all $i \neq 0$.

Recall. If N is non-zero GE, then the shifted module $N(i)$ is not GE for $i \neq 0$.

Techniques used

We say M is strongly generalized Eulerian (SGE) if for all homogeneous element m of M, there exists $a > 0$ such that

$$(\underbrace{\sum_{i=1}^{n} X_i\partial_i}_{\text{Euler opertaor } \mathcal{E}_n} - \deg m)^{a} \cdot m = 0.$$

does not depend on m

Examples. (i) $A_n(K)/J$ if $\mathcal{E}_n^a \in J$ for some $a > 0$, (ii) $H_I^i(R)$.

⋆ We do not have an example of a generalized Eulerian (GE) module which is not SGE.

➢ If M is a f.g. $A_n(K)$-module, then M is GE $\iff$ M is SGE.


Thank you