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Problem Session

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A SHORT LIST OF PROBLEMS FOR THE CONFERENCE: “D-MODULES, GROUP ACTIONS, AND FROBENIUS: COMPUTING ON SINGULARITIES”

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ABSTRACT. This document contains a short list of problems around the general idea of “describing non-reduced schemes with the use of differential operators”.

1. INTRODUCTION

The problem of characterizing ideal membership with differential conditions was first addressed by Gröbner in [11]. He derived such characterizations for ideals that are primary to a rational maximal ideal [12, pages 174-178], and he suggested that the same program could be carried out for any primary ideal [10, §1].

Despite this early algebraic interest by Gröbner, a complete description of primary ideals in terms of differential operators was first obtained by analysts in the *Fundamental Principle of Ehrenpreis and Palamodov* ([8, 14]). At the core of the Fundamental Principle, one has the following theorem by Palamodov.

Theorem 1.1 (Palamodov)**.** *Let R be a polynomial ring $R = \mathbb{C}[x_1,\ldots,x_n]$ over the complex numbers $\mathbb{C}$, $\mathfrak{p} \in \mathrm{Spec}(R)$ be a prime ideal, and $Q \subset R$ be a $\mathfrak{p}$-primary ideal. Then, there exist differential operators $A_1,\ldots,A_m \in R\langle \partial_{x_1},\ldots,\partial_{x_n}\rangle$ such that $Q = \{f \in R \mid A_i \bullet f \in \mathfrak{p} \text{ for } 1 \leqslant i \leqslant m\}$.*

Following the terminology of Palamodov, the differential operators $A_1,\ldots,A_m$ are commonly called *Noetherian operators* for the $\mathfrak{p}$-primary ideal Q. Subsequent algebraic approaches to characterize primary ideals (and, later, arbitrary ideals) with the use of differential operators were given in [1–6, 13]. All the results that are presented subsequently are valid for modules, but for simplicity of notation, we stick to the case of ideals. The theorem below extends Palamodov’s theorem to quite nonrestrictive settings.

Theorem 1.2 ([4])**.** *Let A be a Noetherian domain and R be an A-algebra essentially of finite type. Let $\mathfrak{p} \in \mathrm{Spec}(R)$ be a prime ideal such that $\mathfrak{p} \cap A = 0$, and $Q \subset R$ be a $\mathfrak{p}$-primary ideal. Then, there exist differential operators $A_1,\ldots,A_m \in \mathrm{Diff}_{R/A}(R, R/\mathfrak{p})$ such that $Q = \{f \in R \mid A_i(f) = 0 \text{ for } 1 \leqslant i \leqslant m\}$.*

Additionally, if R is formally smooth over A, then there exist $A_1,\ldots,A_m \in \mathrm{Diff}_{R/A}(R,R)$ such that $Q = \{f \in R \mid A_i(f) \in \mathfrak{p} \text{ for } 1 \leqslant i \leqslant m\}$.

Let $\mathbb{k}$ be a field and R be a $\mathbb{k}$-algebra essentially of finite type. To describe arbitrary ideals instead of just primary ideals the following notion was recently introduced. Our definition rests on localizing along associated prime ideals $\mathfrak{p}_i$, and recovering the localization $I_{\mathfrak{p}_i}$ of the ideal.

Definition 1.3. Let $I \subset R$ be an ideal with $\mathrm{Ass}(R/I) = \{\mathfrak{p}_1,\ldots,\mathfrak{p}_k\} \subset \mathrm{Spec}(R)$. A *differential primary decomposition* of I is a list of pairs $(\mathfrak{p}_1,\mathfrak{A}_1),\ldots,(\mathfrak{p}_k,\mathfrak{A}_k)$, where $\mathfrak{A}_i \subset \mathrm{Diff}_{R/\mathbb{k}}(R, R/\mathfrak{p}_i)$ is a finite set of differential operators, such that the following equation holds for each $\mathfrak{p} \in \mathrm{Ass}(R/I)$:

$$I_{\mathfrak{p}} \;=\; \bigcap_{\substack{1 \leqslant i \leqslant k \\ \mathfrak{p}_i \subseteq \mathfrak{p}}} \left\{ f \in R_{\mathfrak{p}} \mid \delta'(f) = 0 \text{ for all } \delta \in \mathfrak{A}_i \right\}.$$

Here $\delta' \in \mathrm{Diff}_{R_{\mathfrak{p}}/\mathbb{k}}(R_{\mathfrak{p}}, R_{\mathfrak{p}}/\mathfrak{p}_i R_{\mathfrak{p}})$ denotes the localization of an operator $\delta \in \mathfrak{A}_i$.

We have the following notion of multiplicity, which will provide a measure of "complexity from a differential point of view".

Definition 1.4. For an ideal $I \subset R$, its *arithmetic multiplicity* is the positive integer

$$\mathrm{amult}(I) \;:=\; \sum_{\mathfrak{p} \in \mathrm{Ass}(R/I)} \mathrm{length}_{R_{\mathfrak{p}}}\left(H^0_{\mathfrak{p}}(R_{\mathfrak{p}}/IR_{\mathfrak{p}})\right) \;=\; \sum_{\mathfrak{p} \in \mathrm{Ass}(R/I)} \mathrm{length}_{R_{\mathfrak{p}}}\left(\frac{(IR_{\mathfrak{p}} :_{R_{\mathfrak{p}}} (\mathfrak{p}R_{\mathfrak{p}})^{\infty})}{IR_{\mathfrak{p}}}\right).$$

In [15], the length inside the sum was denoted $\mathrm{mult}_I(\mathfrak{p})$ and called the multiplicity of I along $\mathfrak{p}$. It is the length of the largest ideal of finite length in the ring $R_{\mathfrak{p}}/IR_{\mathfrak{p}}$.

The next theorem is the main result regarding differential primary decompositions. An ideal I always has a differential primary decomposition whose total number of operators is equal to the arithmetic multiplicity. Moreover, $\mathrm{amult}(I)$ is a lower bound on the size of any differential primary decomposition.

Theorem 1.5 ([6])**.** *Assume that $\mathbb{k}$ is a perfect field. Fix an ideal $I \subset R$ with $\mathrm{Ass}(R/I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\} \subset \mathrm{Spec}(R)$. The size of a differential primary decomposition is at least $\mathrm{amult}(I)$, and this upper bound is tight. More precisely:*

(i) *I has a differential primary decomposition $(\mathfrak{p}_1, \mathfrak{A}_1), \ldots, (\mathfrak{p}_k, \mathfrak{A}_k)$ such that $|\mathfrak{A}_i| = \mathrm{mult}_I(\mathfrak{p}_i)$.*

(ii) *If $(\mathfrak{p}_1, \mathfrak{A}_1), \ldots, (\mathfrak{p}_k, \mathfrak{A}_k)$ is a differential primary decomposition for I, then $|\mathfrak{A}_i| \geqslant \mathrm{mult}_I(\mathfrak{p}_i)$.*

In this document, we describe some problems in a similar vein to the above results.

2. Some problems

(1) Implementation of an algorithm to compute differential primary decompositions in the positive characteristic case. Right now, in `Macaulay2` [3,9] one can compute a differential primary decomposition for any ideal (actually, for any submodule of a free module) in a polynomial ring over a field of characteristic zero (as proposed in [6]). Therefore, the question is: *to implement the analog algorithm in the positive characteristic case*. The implementation will be quite a bit more cumbersome than in the characteristic zero case, but in principle all techniques should be easily adaptable. Below, to wit, we have a Macaulay2 session where a differential primary decomposition is computed for a module.

Example 2.1 ([2, Example 6.2])**.** Let $R = \mathbb{Q}[x_1, x_2, x_3]$ and $U \subseteq R^2$ be the R-submodule

$$U = \mathrm{image}_R \begin{bmatrix} x_1^2 & x_1x_2 & x_1x_3 \\ x_2^2 & x_2x_3 & x_3^2 \end{bmatrix}.$$

We compute a primary decomposition and a minimal differential primary decomposition for U:

```
Macaulay2, version 1.17.2.1
i1 : load "modulesNoetherianOperators.m2";
i2 : printPD = M -> apply(primaryDecomposition M, Q -> trim image(gens Q | relations Q));
i3 : R = QQ[x_1,x_2,x_3];
```

```
i4 : U = image matrix {{x_1^2,x_1*x_2,x_1*x_3}, {x_2^2,x_2*x_3,x_3^2}};
i5 : M = R^2 / U;
i6 : L1 = printPD M
o6 = {image | 0 x_1 |, image | x_1 x_2^2 0             |, image | x_3 x_2^2 0      x_1x_2 x_1^2 |}
            | 1 0   |        | x_3 x_3^2 x_2^2-x_1x_3 |         | 0   0     x_3^2 x_2x_3 x_2^2 |
o7 : all(L1, isPrimary_M) and U == intersect L1
o7 = true
i8 : L2 = differentialPrimaryDecomposition U
2
o8 = {{ideal x , {| 1 |}}, {ideal(x  - x x ), {| -x_3 |}}, {ideal (x , x ), {| 0    |}}}
              1   | 0 |             2    1 3    | x_1  |               3   2    | dx_3 |
o8 : List
i9 : U == intersect apply(L2, getModuleFromNoetherianOperators)
o9 = true
i10 : amult U
o10 = 3
```

Notice that $\mathrm{amult}(U) = 3$ is the size of the computed differential primary decomposition.

(2) Computing a differential primary decomposition for your favorite family of ideals. The question is: *to choose a nice family of ideals and give an explicit description of a minimal differential primary decomposition*. Nice examples could be: edge ideals, binomial edge ideals, monomial ideals, toric ideals or ideals associated to subspace arrangements ([7]). Another example, already solved in [6, Theorem 7.1], [2, Theorem 5.3], is the characterization of primary ideals coming from the join construction as ideals that can be described with differential operators with constant coefficients.

(3) Drop the perfect field assumption in Theorem 1.5. Note that an ideal always has a differential primary decomposition by invoking Theorem 1.2 on the components of a primary decomposition (see [6, Remark 3.4]). So, the important question is: *"what is the minimum size of a differential primary decomposition?"*. When the field $\mathbb{k}$ is not perfect, then the minimum size can be higher that the arithmetic multiplicity (see [4, Example 4.8]). For a non perfect field $\mathbb{k}$, the minimum size should be a lot less clean than in Theorem 1.5, as it should be related to the inseparable degree of the residue fields of the associated primes over $\mathbb{k}$.

(4) Extending the representation theorem of [5]. The paper [5] contains a "representation theorem" that characterizes primary ideals via three different but closely related objects (see [6, Theorem 2.1]). Of particular interest is the parametrization of primary ideal with the use of *punctual Hilbert schemes*. These results were extended afterward for the case of modules in [2]. The question is: *to obtain a parameter space of an arbitrary ideal* I *with the fixed multiplicities* $\mathrm{mult}_I(\mathfrak{p}_i)$ *along the associated primes* $\mathrm{Ass}(R/I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$ *of* I. For the case of embedded associated primes, the answer (if it exists) should be some sort of *flag Hilbert schemes*. Of course, the case where there is no embedded associated prime the answer follows directly from [6, Theorem 2.1].

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D-modules, group actions, and Frobenius: computing on singularities

Lengths of D-modules on singularities

1) Let R be a polynomial ring over a field of char. 0, and G be a finite group acting on R.

When do equalities hold:

- $\ell_{D_{R^G}}(R^G_f) = \ell_{D_R}(R_f)$ for $f \in R^G$,

- $\ell_{D_{R^G}}(H^i_I(R^G)) = \ell_{D_R}(H^i_{IR}(R))$ for $I \subseteq R^G$,

in terms of f, I, G?

In general, "$\leq$" holds:

<u>Def</u> (Àlvarez Montaner, Huneke, Núñez-Betancourt):

Let $A \subseteq B$ be an inclusion of (commutative) K-algebras that splits as A-modules via $\beta: B \to A$.

A $D_{A|K}$-module M is a <u>differential direct summand</u> of a $D_{B|K}$-module N if $M \subseteq N$ with splitting $\theta: N \to M$ as abelian groups such that

for all $m \in M$, $\delta \in D_{B|K}$,

$$\theta(\underbrace{\delta \cdot m}_{D_B\text{-action}}) = \underbrace{(\beta \circ \delta|_A) \cdot m}_{D_A\text{-action}}.$$

Thm (AHN): If M is a differential direct summand of N, then

$$\ell_{D_A}(M) \le \ell_{D_B}(N).$$

R^G_f is a differential direct summand of R_f

$$H^i_I(R^G) \text{ --------- } H^i_{IR}(R)$$

Thm (AHJNTW): M is a holo. D_{R^G}-mod $\iff$ M is a diff'l direct summand of some holo D_R-mod.

Possible starting cases:

- $R^G = \mathbb{C}[x^2, y^2] \subseteq R = \mathbb{C}[x, y]$
- $R^G = \mathbb{C}[x^2, xy, y^2] \subseteq R = \mathbb{C}[x, y]$

↳ Does R^G_f/R simple $/D_{R^G} \Rightarrow R_f/R$ simple $/D_R$?

2) Let R be a strongly F-regular graded K-algebra with finite F-representation type (in char $= p > 0$).

Give upper bounds on the length of R_f in terms of f and numerical invariants of R by using filtrations and multiplicity.

ON THE BERNSTEIN–SATO POLYNOMIALS OF ROOTS OF POLYNOMIALS

Think of the $n+1$-dimensional affine space ($n > 1$) as $X = \mathrm{Sym}^n W$ with $\dim W = 2$. We let x, y be a basis of W, and $x^n, x^{n-1}y, \dots, xy^{n-1}, y^n$ a basis of X, with respective coordinates $x_n, x_{n-1}, \dots, x_1, x_0$, so that we identify $\mathbb{C}[X] = \mathbb{C}[x_0, \dots, x_n]$.

We denote by $r = r(x_0, \dots, x_n)$ an algebraic function (on some domain in X) that satisfies

$$x_n \cdot r^n + x_{n-1} \cdot r^{n-1} + \cdots + x_1 \cdot r + x_0 = 0.$$

Throughout $f \in \mathbb{C}[X]$ denotes the discriminant of the polynomial above, with $\deg f = 2n - 2$.

Let $\mathfrak{g}$ be the Lie algebra of $G = \mathrm{GL}(W)$, $U\mathfrak{g}$ its universal enveloping algebra. The natural action of G on X gives the Lie algebra map $\mathfrak{g} \to \mathcal{D}_X$, and we pick the following basis of vector fields for its image:

$$g_{11} = x_1\partial_1 + 2x_2\partial_2 + \cdots + nx_n\partial_n, \quad g_{12} = nx_0\partial_1 + (n-1)x_1\partial_2 + \cdots + x_{n-1}\partial_n,$$
$$g_{21} = x_1\partial_0 + 2x_2\partial_1 + \cdots + nx_n\partial_{n-1}, \quad g_{22} = nx_0\partial_0 + (n-1)x_1\partial_1 + \cdots + x_{n-1}\partial_{n-1}.$$

Problem. The goal is to find the Bernstein–Sato polynomial of the algebraic function $h = x_{n-1} + nx_n \cdot r$.

(a) Show that h is G-finite (i.e. $U\mathfrak{g} \cdot h$ is a finite-dimensional G-module), so that $\mathcal{D}_X \cdot h$ is a G-equivariant $\mathcal{D}_X$-module.

(b) [K. Mayr '37] Show that the following operators annihilate r:

$$g_{11} + 1,\ g_{22} - 1, \quad \partial_i\partial_j - \partial_k\partial_l \ \text{ with } i + j = k + l.$$

(c) Prove that the following operators generate the annihilating ideal of h in $\mathcal{D}_X$:

$$g_{11} - n + 1,\ g_{12}^{n-1},\ g_{21},\ g_{22} - 1, (\partial_i\partial_{j+1} - \partial_{i+1}\partial_j)^{1+\delta_{j,n-1}}, \ \text{ for } 0 \le i < j \le n-1 \ (\delta \text{ is the Kronecker delta}).$$

In fact, $\mathcal{D}_X \cdot h$ is a simple (regular, holonomic) $\mathcal{D}_X$-module and its Fourier transform is isomorphic to the local cohomology module $H_Z^{n-1}(\mathbb{C}[X])$, where $Z \subset X$ is the Veronese cone.

(d) Show that the singular locus of $\mathcal{D}_X \cdot r$ is defined by $x_n \cdot f$, whereas that of $\mathcal{D}_X \cdot h$ by f.

(e) Compute the Bernstein–Sato polynomial $b_h(s)$ of h.

Recall that the Bernstein–Sato polynomial of h is the minimal monic polynomial $b_h(s) \in \mathbb{C}[s]$ for which there exists an operator $P \in \mathcal{D}_X[s]$ such that

$$P \cdot f^{s+1}h = b_h(s) \cdot f^s h.$$

It is known that the roots of $b_h(s)$ are rational, and that -1 and $-3/2$ are roots of $b_h(s)$ (for $n > 2$).

A FEW QUESTIONS

MIRCEA MUSTAŢĂ

1. Questions about b-functions

Let X be a smooth, n-dimensional complex algebraic variety, $P \in X$ a point, and $f_1, \ldots, f_r \in \mathcal{O}_X(X)$ nonzero, with $f_i(P) = 0$. Let $f = \prod_{i=1}^r f_i$ and we consider the Bernstein-Sato polynomial $b_{f,P}(s)$ of f at P. We assume that $f_1, \ldots, f_r$ form a regular sequence at P.

Fact: One can show that under the above assumptions, $(s+1)^r$ divides $b_{f,P}(s)$.

Definition 1.1. We define $\widetilde{\alpha}_P(f_1, \ldots, f_r)$ to be equal to the negative of the largest root of $b_{f,P}(s)/(s+1)^r$. Note that this is equal to $\mathrm{lct}_P(f)$ if $\mathrm{lct}_P(f) < 1$.

Remark 1.2. Note that if $r = 1$, then we recover the *minimal exponent* of f at P.

Remark 1.3. One can show that if $f_1, \ldots, f_r$ do not form a regular sequence at P, then $\mathrm{lct}_P(f) < 1$. Therefore it is natural to put $\widetilde{\alpha}_P(f_1, \ldots, f_r) = \mathrm{lct}_P(f)$ in this case.

Question 1.4. Suppose that $f_1, \ldots, f_r \in \mathbf{C}[x_1, \ldots, x_n]$ are homogeneous, with $\deg(f_i) = d_i$, defining a smooth complete intersection subvariety of $\mathbf{P}^{n-1}$. What is $\widetilde{\alpha}_P(f_1, \ldots, f_r)$?

The following are some more theoretical questions:

Question 1.5. Suppose that $X = X_1 \times \ldots \times X_r$, $P = (P_1, \ldots, P_r)$, and $f_i = g_i \circ \mathrm{pr}_i$, where $g_i \in \mathcal{O}_{X_i}(X_i)$. Is it true that $b_{f,P}(s) = \prod_{i=1}^r b_{g_i,P_i}(s)$? This would imply

$$\widetilde{\alpha}_P(f_1, \ldots, f_r) = \min_i \widetilde{\alpha}_{P_i}(g_i).$$

It is certainly clear that $b_{f,P}(s)$ divides $\prod_{i=1}^r b_{g_i,P_i}(s)$ and thus $\widetilde{\alpha}_P(f_1, \ldots, f_r) \geq \min_i \widetilde{\alpha}_{P_i}(g_i)$.

Question 1.6. By definition of $\widetilde{\alpha}_P(f_1, \ldots, f_r)$, this is infinite precisely when we have $b_{f,P}(s) = (s+1)^r$. For example, this is the case if the divisors $V(f_i)$ are smooth and intersect transversely. Does the converse hold, that is, if $b_{f,P}(s) = (s+1)^r$, can we say that in some neighborhood of P, the divisors $V(f_i)$ are smooth and intersect transversely?

Question 1.7. Is it true that if $\widetilde{\alpha}_P(f_1, \ldots, f_r) > 1$, then $V(f_1, \ldots, f_r)$ has rational singularities at P? Is the converse true: if $V(f_1, \ldots, f_r)$ has rational singularities at P, do we have $\widetilde{\alpha}_P(f_1, \ldots, f_r) > 1$ if we replace $f_1, \ldots, f_r$ by general linear combinations?

2. A question about the Briançon-Skoda theorem

Let X be a smooth n-dimensional algebraic variety over an algebraically closed field k. Let $f \in \mathcal{O}_X(X)$ be nonzero and J_f the Jacobian ideal of f (if on $U \subseteq X$ open we have local algebraic coordinates $x_1, \ldots, x_n$, then J_f is generated in U by $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$).

A well-known result due to Briançon-Skoda (for $k = \mathbf{C}$) and to Lipman-Teissier in the general case says that $f^n \in J_f$. In the recent preprint [JKSY21] one gives an improvement, when $k = \mathbf{C}$, in terms of the minimal exponent $\widetilde{\alpha}(f)$: it is shown that $f^k \in J_f$ if $k \geq n - \lceil \widetilde{2\alpha}(f) \rceil + 1$. For example, if $\mathrm{lct}(f) > \frac{1}{2}$, then $f^{n-1} \in J_f$ and if the hypersurface defined by f has rational singularities, then $f^{n-2} \in J_f$.

Question 2.1. Suppose now that $\mathrm{char}(k) = p > 0$.

i) iI $\mathrm{fpt}(f) > \frac{1}{2}$, do we have $f^{n-1} \in J_f$?

ii) If the hypersurface defined by f has F-rational singularities, do we have $f^{n-2} \in J_f$?

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Jeffries, S.: Let A be $\mathbb{Z}$ or $\hat{\mathbb{Z}}_{(p)}$ and R be any of:

① $A[X]/\det X$ $\quad X : n \times n$ where $n \geq 3$

② $A[X]/\mathrm{pfaff}\, X$ $\quad X : n \times n$ alt, $n \geq 4$ even

③ $A[X]/\det X$ $\quad X : n \times n$ sym, $\begin{cases} n \geq 4 \\ \text{OR} \\ n = 3,\ p = 2 \end{cases}$

Then R is <u>not</u> a direct summand of a polynomial ring $/A$.

<u>Idea</u>: Frobenius lifting; criterion of Zdanowicz

Suppose p is a prime integer that is not a unit in a ring S. A lift of the Frobenius endomorphism F of S/pS is an endomorphism Λ_p of S such that

$$\begin{tikzcd} S \arrow[r, "\Lambda_p"] \arrow[d] & S \arrow[d] \\ S/pS \arrow[r, "F"] & S/pS \end{tikzcd}$$

commutes.

If $R \hookrightarrow S$ is R-split then

S/p^2S has a Frob lift $\Longrightarrow$ R/p^2R has a Frob. lift.

Hankel matrix of indeterminates

$$H := \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & \cdots & & x_s \\ x_2 & x_3 & x_4 & \cdots & & & \\ x_3 & x_4 & \cdots & & & & \\ x_4 & & & & & & \\ \vdots & & & & & & \\ x_r & x_{r+1} & & \cdots & & & x_{r+s-1} \end{pmatrix}$$

Hankel determinantal ring over A :

$$R_A := A[H]/I_t(H)$$

Question: Is $R_{\mathbb{C}}$ a direct summand of a polynomial ring over $\mathbb{C}$?

Set $A := \widehat{\mathbb{Z}}_{(p)}$, p a prime integer.

<u>Question</u>: If H is a $t \times t$ Hankel matrix of indeterminates and $R_A := A[H]/\det H$, is R_A a direct summand of a polynomial ring over A?

<u>$t=2$</u> $\quad R_A = \dfrac{A\begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}}{x_2^2 - x_1 x_3} \cong A[s^2, st, t^2] \hookrightarrow A[s,t]$

$\boxed{t \geq 3}$ Prove, if possible, that $R_A/p^2 R_A$ does not have a Frobenius lift.

Set $f := \det M$, so that $R_A := A[\underline{x}]/(f)$

$$\varphi_p(f) := \left(f(x_1^p, \dots, x_{2t-1}^p) - f^p \right) / p$$

Zdanowicz: The Frobenius endomorphism on R_A/pR_A lifts to R_A/p^2R_A if and only if

$$\varphi_p(f) \in \left(p, f, \left(\frac{\partial f}{\partial x_i}\right)^p : 1 \leq i \leq 2t-1 \right) R_A.$$