Hearing the shape of a room

an unlabeled distance geometry problem

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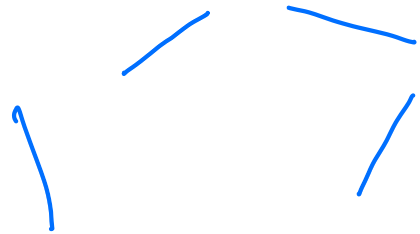
Joint work with Gregor Kemper, TU Munich.

1. Hearing the Shape of a Room



Given is a *room*:

- arrangement of planar *"walls"* (ceilings, floors, . . .);
- not necessarily convex;
- not necessarily closed;
- position and number of walls is unknown.

We want to use sound to determine the walls.

mapping vs location

1. Hearing the Shape of a Room

We use 4 microphones:

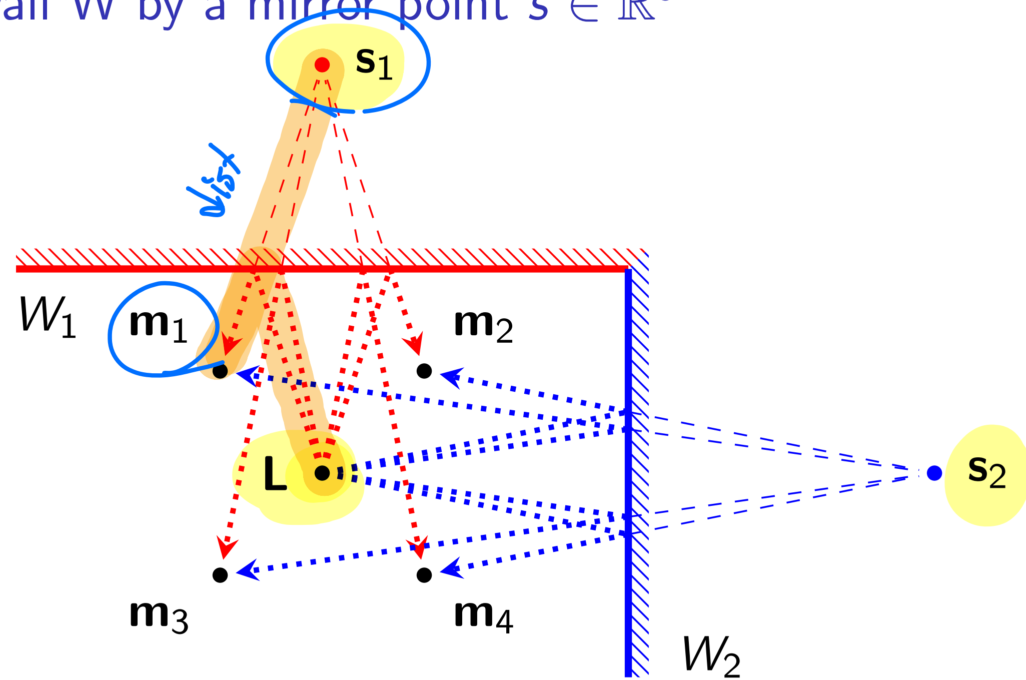
- known positions $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3, \mathbf{m}_4 \in \mathbb{R}^3$;
- placed on a drone.

An omnidirectional speaker emits a short pulse:

- high-frequency so ray acoustics approximation holds;
- 1^{st} order echoes: pulses heard after they bounce off the walls;
- 2^{nd} order echoes: bounced off pulses bounce again;
- etc.

1. Hearing the Shape of a Room

Represent each wall W by a mirror point $s \in \mathbb{R}^3$



Virtually, sound comes from mirror points $\mathbf{s}_1$ and $\mathbf{s}_2$.

Suppose microphones are at positions $\mathbf{m}_i \in \mathbb{R}^3$, $i = 1, 2, 3, 4$.

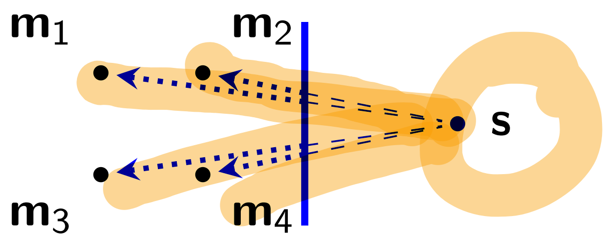
Fact 1

The microphones are coplanar if and only if $\det(M) = 0$ with

$$M := \begin{pmatrix} \mathbf{m}_1 & \mathbf{m}_2 & \mathbf{m}_3 & \mathbf{m}_4 \\ \hline 1 & 1 & 1 & 1 \end{pmatrix} \in \mathbb{R}^{4\times 4}.$$

1. Hearing the Shape of a Room

Given 4 non-planar microphones $\mathbf{m}_i \in \mathbb{R}^3$ and a wall $\mathbf{s} \in \mathbb{R}^3$.



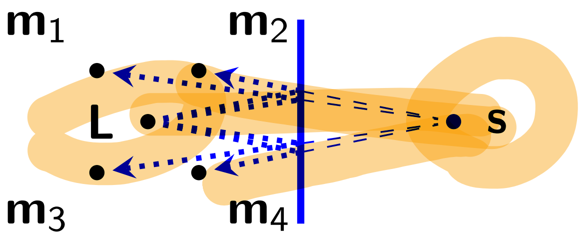
Fact 2

- Let $\tilde{M} \in \mathbb{R}^{3\times 4}$ be the upper 3×4-part of $(M^{-1})^T$.
- $\mathbf{s}$ can be computed from the squared distances $d_i := \|\mathbf{s} - \mathbf{m}_i\|^2$ as

$$\mathbf{s} = \frac{1}{2}\tilde{M} \cdot \begin{pmatrix} \|\mathbf{m}_1\|^2 - d_1 \\ \vdots \\ \|\mathbf{m}_4\|^2 - d_4 \end{pmatrix}.$$

1. Hearing the Shape of a Room

Let $\mathbf{L} \in \mathbb{R}^3$ be the position of the loudspeaker.



Fact 3

- Wall $\mathbf{s}$ has normal vector $\mathbf{s} - \mathbf{L}$ and contains point $\frac{1}{2}(\mathbf{s} + \mathbf{L})$.
- Four non-collinear points on the wall are found by intersecting line between $\mathbf{s}$ and $\mathbf{m}_i$ $(1 \leq i \leq 4)$ with this plane.
- Points given by

$$(1 - \tau_i)\mathbf{s} + \tau_i \mathbf{m}_i \quad \text{with} \quad \tau_i = \frac{\|\mathbf{s} - \mathbf{L}\|^2}{2\langle \mathbf{s} - \mathbf{L}, \mathbf{s} - \mathbf{m}_i \rangle}.$$

Summary: if room has only one wall $\mathbf{s}$

- Time between pulse emitted and first order echoes gives microphone-wall distance

$$d_i = \|m_i - \mathbf{s}\|^2.$$

- Reconstruct $\mathbf{s}$ from $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3, \mathbf{m}_4$ and d_1, d_2, d_3, d_4.
- Reconstruct wall normal from $\mathbf{s}$ and $\mathbf{L}$.
- Reconstruct 4 points on wall by intersecting wall plane with line from $\mathbf{s}$ to $\mathbf{m}_i$, $i = 1, \ldots, 4$.

If room has several walls s_j

- Each microphone $\mathbf{m}_i$ receives echoes from K_i walls.
- Distance to each of the walls computed from elapsed time.
- Let $\mathcal{D}_i = \{\|\mathbf{m}_i - \mathbf{s}_j\|^2\}_{j=1}^{K_i}$, be the multiset of (unlabeled) distances from m_i to the walls it hears. Set

$$\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2 \times \mathcal{D}_3 \times \mathcal{D}_4.$$

- Want to reconstruct the walls from $\mathcal{D}$.

The challenge: sorting the echoes

($=$ labeling the distances)

- Determining whether four distances $(d_1, d_2, d_3, d_4) \in \mathcal{D}$ could correspond to one wall,
- i.e., determine if there exists a wall $\mathbf{s}$ s.t.

$$\|\mathbf{m}_i - \mathbf{s}\|^2 = d_i, \quad \text{for } i = 1, 2, 3, 4.$$

Five Point Echo Sorting Criterion[1]:

Let $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3, \mathbf{m}_4, \mathbf{m}_5 \in \mathbb{R}^3$.
Let $D_{i,j} = \|\mathbf{m}_i - \mathbf{m}_j\|^2$, $i, j = 1, 2, 3, 4, 5$ and $u_1, \ldots, u_5 \in \mathbb{R}$. Let

$$E := \begin{pmatrix} 0 & u_1 & \cdots & u_5 \\ u_1 & D_{1,1} & \cdots & D_{1,5} \\ \vdots & \vdots & & \vdots \\ u_4 & D_{4,1} & \cdots & D_{4,5} \\ u_5 & D_{5,1} & \cdots & D_{5,5} \end{pmatrix} \quad \text{and} \quad g_E(u_1, u_2, \ldots, u_5) := \det(E).$$

Then $g_E(d_1, d_2, d_3, d_4, d_5) = 0$ when d_1, d_2, d_3, d_4, d_5 correspond to same wall $\mathbf{s}$.

[1] Dokmanić et al., "Acoustic echoes reveal room shape".

2. Echo Sorting Criterion

Why?

Because if $x_1, \ldots, x_k \in \mathbb{R}^n$, then the Euclidean Distance Matrix

$$\begin{pmatrix} \|x_1 - x_1\|^2 & \|x_1 - x_2\|^2 & \cdots & \|x_1 - x_k\|^2 \\ \|x_2 - x_1\|^2 & \|x_2 - x_2\|^2 & \cdots & \|x_2 - x_k\|^2 \\ \|x_3 - x_1\|^2 & \|x_3 - x_2\|^2 & \cdots & \|x_3 - x_k\|^2 \\ \vdots & \vdots & & \vdots \\ \|x_k - x_1\|^2 & \|x_k - x_2\|^2 & \cdots & \|x_k - x_k\|^2 \end{pmatrix} \in \mathbb{R}^{k \times k}$$

has rank at most $n + 2$.

Four Microphone Echo Sorting Criterion:

Let $D_{i,j} := \|\mathbf{m}_i - \mathbf{m}_j\|^2$ and let $u_1 \dots u_4 \in \mathbb{R}$. Let

$$D := \begin{pmatrix} 0 & u_1 & \cdots & u_4 & 1 \\ u_1 & D_{1,1} & \cdots & D_{1,4} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ u_4 & D_{4,1} & \cdots & D_{4,4} & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix} \quad \text{and} \quad f_M(u_1 \dots u_4) := \det(D). \tag{1}$$

Then $f_M(d_1 \dots d_4) = 0$ when d_1, d_2, d_3, d_4 correspond to the same wall $\mathbf{s}$.

2. Echo Sorting Criterion

Why?

- Set $\mathbf{m}_0 = \mathbf{s}$.
- Then

$$\det D = \det \begin{pmatrix} D_{0,0} & D_{0,1} & \cdots & D_{0,4} & 1 \\ D_{1,0} & D_{1,1} & \cdots & D_{1,4} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ D_{4,0} & D_{4,1} & \cdots & D_{4,4} & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}$$

is the Cayley-Menger determinant of the 5-simplex $\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3, \mathbf{m}_4$.

The Algorithm

1. For $i = 1, \ldots, 4$, collect times of echoes recorded by the ith microphone in set $\mathcal{T}_i$.
2. Set $\mathcal{D}_i := \{c^2(t - t_0)^2 \mid t \in \mathcal{T}_i\}$ $(i = 1, \ldots, 4)$, where c speed of sound, t_0 time of sound emission.
3. FOR $(d_1, d_2, d_3, d_4) \in \mathcal{D}_1 \times \mathcal{D}_2 \times \mathcal{D}_3 \times \mathcal{D}_4$ DO
   - 3.4 IF $f_M(d_1, \ldots, d_4) = 0$ THEN
     - 3.4.5 Compute mirror point **s** from $(d_1, \ldots, d_4)$.
     - 3.4.6 Compute four non-collinear points on the wall with mirror point **s**
     - 3.4.7 OUTPUT data of this wall.

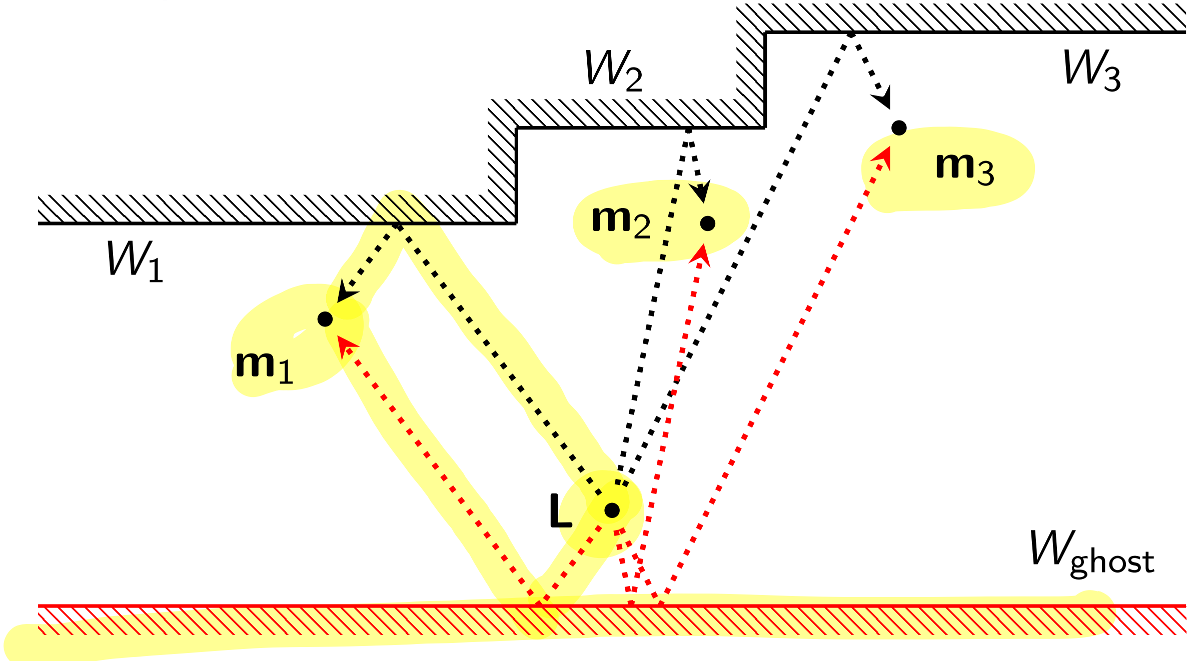
3. Wall Reconstruction Algorithm

Observe:

- Algorithm reconstructs all walls heard by 4 microphones.
- Algorithm could reconstruct walls that are not there (ghost walls).

3. Wall Reconstruction Algorithm

Example of Ghost wall



3. Wall Reconstruction Algorithm

How to prevent ghost walls?

Case of Generic Walls

- Polynomial $f_M(u_1, u_2, u_3, u_4)$ unchanged when moving walls.
- If $f_M(\|m_1 - s_1\|^2, \|m_2 - s_2\|^2, \|m_3 - s_3\|^2, \|m_4 - s_4\|^2) = 0$ with s_1, s_2, s_3, s_4 not all equal, then move walls s_i's slightly to avoid zero set.
- After moving walls, algorithm will only reconstruct walls that are there.

So can hear shape of room with walls in generic positions.

3. Wall Reconstruction Algorithm

Want to consider potentially non-generic wall configurations

e.g., parallel/perpendicular walls.

Can we move the drone slightly instead of the walls?

Definition

Microphones (or drone) are in *good position* if wall detection algorithm detects no ghost walls; else they are in *bad Position*.

Theorem (B.-Kemper)

The set of bad drone positions lies in a subspace of dimension ≤ 5 within the 6-dimensional space of possible drone positions.

$\Rightarrow$ If drone in generic position, our algorithm only reconstructs walls that are there.

Conclusion

A drone in generic position can hear the shape of a room from echoes!

4. Related Problem: Shape from Pairwise Distances

Given the pairwise distances

$$\left\{\|x_i - x_j\|^2\right\}_{i,j=1}^{k}$$

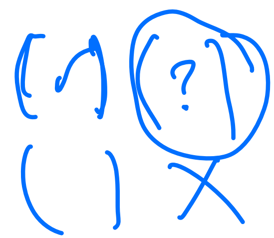
can we reconstruct the points $x_1, \ldots, x_k \in \mathbb{R}^n$?

The labeled case

- Reconstruct $x_1, \dots, x_k \in \mathbb{R}^n$ from labeled distances $d_{ij} = \|x_i - x_j\|^2$.

Here, matrix of pairwise distances is known

$$\begin{pmatrix} \|x_1 - x_1\|^2 & \|x_1 - x_2\|^2 & \cdots & \|x_1 - x_k\|^2 \\ \|x_2 - x_1\|^2 & \|x_2 - x_2\|^2 & \cdots & \|x_2 - x_k\|^2 \\ \|x_3 - x_1\|^2 & \|x_3 - x_2\|^2 & \cdots & \|x_3 - x_k\|^2 \\ \vdots & \vdots & & \vdots \\ \|x_k - x_1\|^2 & \|x_k - x_2\|^2 & \cdots & \|x_k - x_k\|^2 \end{pmatrix} \in \mathbb{R}^{k \times k}$$



4. Related Problem: Shape from Pairwise Distances

The labeled case

- Reconstruct $x_1, \ldots, x_k \in \mathbb{R}^n$ from labeled distances $d_{ij} = \|x_i - x_j\|^2$.

Solution

- Matrix $\Delta = \left(\|x_i - x_k\|^2 - \|x_j - x_k\|^2 - \|x_i - x_j\|^2\right)$ factors as

$$\Delta = (x_i - x_k)^T (x_j - x_k).$$

- singular value decomposition of $\Delta = Q^T \Sigma Q$ yields solution

$$\begin{pmatrix} x_i - x_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \sqrt{\Sigma} Q$$

4. Related Problem: Shape from Pairwise Distances

distribution of distances
bag of distances

The unlabeled case

- Reconstruct $x_1, \ldots, x_k \in \mathbb{R}^n$ from multiset of distances $\{d_{ij} = \|x_i - x_j\|^2\}$.

Here, matrix of pairwise distances is known up to a shuffling

$$\begin{pmatrix} \|x_1 - x_1\|^2 & \|x_1 - x_2\|^2 & \cdots & \|x_1 - x_k\|^2 \\ \|x_2 - x_1\|^2 & \|x_2 - x_2\|^2 & \cdots & \|x_2 - x_k\|^2 \\ \|x_3 - x_1\|^2 & \|x_3 - x_2\|^2 & \cdots & \|x_3 - x_k\|^2 \\ \vdots & \vdots & & \vdots \\ \|x_k - x_1\|^2 & \|x_k - x_2\|^2 & \cdots & \|x_k - x_k\|^2 \end{pmatrix} \in \mathbb{R}^{k \times k}$$

4. Related Problem: Shape from Pairwise Distances

Reconstructing a point configuration from unlabeled distances is also a problem encountered in

- x-ray crystallography (Patterson 1935, Patterson 1944)
- mapping of restriction sites of DNA- *partial digest problem*- (Stefik 1978, Dix and Kieronska 1988, Gwangsoo 1988,...)
- material science (Jiao-Stillinger-Torquato 2010)
- ...

"Turnpike Problem" or "Partial Digest Problem": points lie in $\mathbb{R}$.
"Beltway Problem": the points lie on a circle.

4. Related Problem: Shape from Pairwise Distances

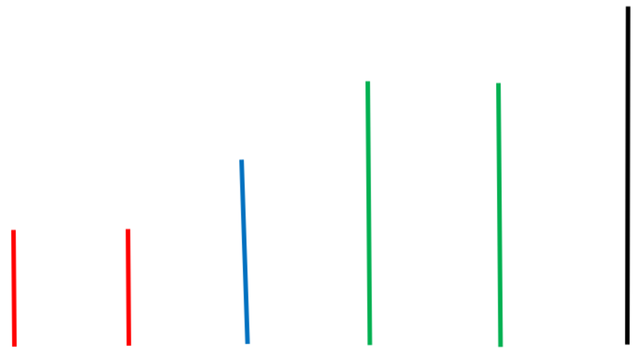
Question: Is the problem well-posed?

i.e., is the shape of a point-set *uniquely* determined by its (unlabeled) pairwise distances?

Example

Is there a unique configuration of 4 points in the plane (up to a rigid motion) whose pairwise distances are

$$\{\sqrt{2}, \sqrt{2}, 2, \sqrt{10}, \sqrt{10}, 4\}?$$

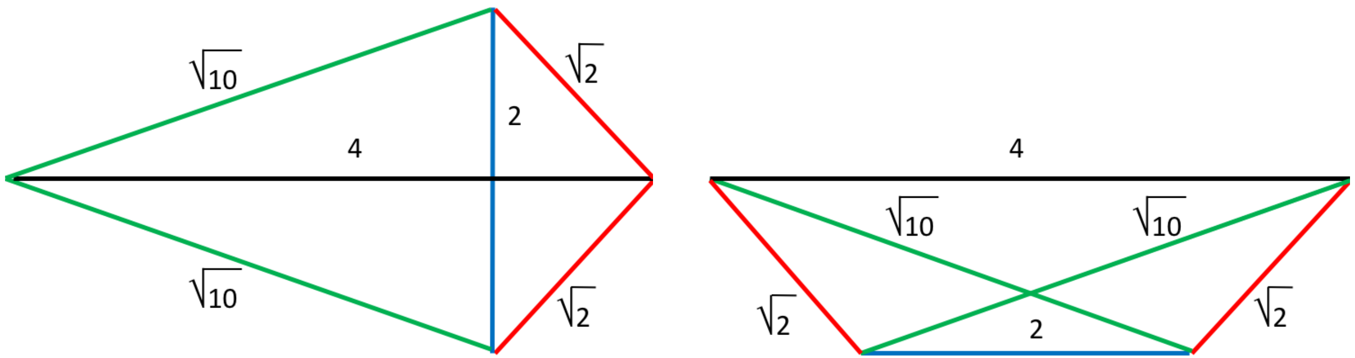


4. Related Problem: Shape from Pairwise Distances

Question: Is the problem well-posed?

No.

Counterexample



Two point-sets with the same pairwise distances

4. Related Problem: Shape from Pairwise Distances

Question: Is the problem well-posed?

For Turnpike Problem ($D = 1$):

- Picard (1939): Proof of uniqueness when no repeated distances.
- Bloom (1977): 6-point counterexample.

4. Related Problem: Shape from Pairwise Distances

Theorem (B.-Kemper)

Let $k \in \mathbb{N}$ with $0 < k \leq 3$ or $k \geq n + 2$
There exists a non-zero polynomial in nk variables such that every k-point configuration $p_1, \ldots, p_k \in \mathbb{R}^n$ with $f(p_1, \ldots, p_k) \neq 0$ is uniquely determined, up to a rigid motion, by the multiset of its unlabeled pairwise distances.

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Proof

- The $(n+1) \times (n+1)$ minors of the matrix $\Delta = \left(\|x_i - x_k\|^2 - \|x_j - x_k\|^2 - \|x_i - x_j\|^2\right)$ are zero.
- The ideal I of syzygies between the distances $D_{i,j}$ is generated by those minors.
- Show that I is not preserved under distance label permutations that do not correspond to point label permutation.

4. Related Problem: Shape from Pairwise Distances

Theorem (B.-Kemper)

Let $n \in \mathbb{N}$ with $0 < n \leq 3$ or $n \geq m + 2$
There exists a non-zero polynomial in mn variables such that every n-point configuration $p_1, \ldots, p_n \in \mathbb{R}^m$ with $f(p_1, \ldots, p_n) \neq 0$ is uniquely determined, up to a rigid motion, by the multiset of its unlabeled pairwise distances.

Corollary

- The set of exceptional point configurations has measure zero.
- Fast comparison algorithm that is accurate with probability 1.

Theorem (B.-Kemper)

Let G be a graph with $n > 5$ nodes and generic real-valued edge weights $g_{i,j} \in \mathbb{R}$. Then G is reconstructible, up to a graph isomorphism, from the following two multi-sets:

$$\{g_{i,j} | i, j \text{ are distinct}\}$$

$$\{g_{i,j} + g_{j,k} | i, j, k \text{ are distinct }\}$$

Thank you!

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References

- M. Boutin, G. Kemper, "A Drone Can Hear the Shape of a Room," to appear in *SIAM Journal on Applied Algebra and Geometry* (2019).
- M. Boutin and G. Kemper, "Lossless Graph Representation using Distributions," https://arxiv.org/abs/0710.1870.
- M. Boutin and G. Kemper, "Which Point Configurations are Reconstructible from the Distribution of their Pairwise Distances?," *International Journal of Computational Geometry and Applications*, Vol. 17, No. 1, pp. 31-43, 2007.
- M. Boutin and G. Kemper, "On Reconstructing Configurations of Points in P2 from a Joint Distribution of Invariants," *Journal of Applicable Algebra In Engineering, Communication and Computing*, Vol. 15, No. 6, pp. 361-391, 2005.
- M. Boutin and G. Kemper, "On Reconstructing n-point Configurations from the Distribution of Distances of Areas," *Advances in Applied Mathematics*, Vol. 32, pp. 709-735, 2004.