Bernstein's inequality & holonomicity for certain singular rings

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D-modules, group actions, and Frobenius: computing on singularities

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I. Classical Bernstein inequality & consequences

For now,

$R = \mathbb{C}[x_1, \dots, x_d]$ poly ring of dimension d

$D_R = R\langle \partial_1, \dots, \partial_d \rangle$ ring of ~~differential~~ operators

$D_R^i = \bigoplus_{|\alpha| \le i} R \cdot \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$ operators of order $\le i$

$\mathrm{gr}^{\mathrm{ord}}(D_R) \simeq R[\xi_1, \dots, \xi_d]$ ~~commutative~~ polynomial ring dimension of a D_R-module

$\mathcal{M} = D_R \cdot \{m_1, \dots, m_t\}$ f.g. D_R-module

$$\mathcal{M} \rightsquigarrow \mathrm{gr}^{\mathrm{ord}}(\mathcal{M}) = \bigoplus \frac{D_R^i\{m_1,\dots,m_t\}}{D_R^{i-1}\{m_1,\dots,m_t\}}$$

f.g. $\mathrm{gr}^{\mathrm{ord}}(D_R)$-module

$\rightsquigarrow$ usual notion of dimension of a module

$$\rightsquigarrow \dim(\mathcal{M}) = \dim_{\mathrm{gr}^{\mathrm{ord}}(D_R)}\left(\mathrm{gr}^{\mathrm{ord}}(\mathcal{M})\right)$$

Bernstein's inequality

Theorem [B≥]: If M is a nonzero fin. gen. D_R-module, then
$$\dim(M) \geq d \;(= \dim(R))$$

A **holonomic** D_R-module is a D_R-module with dimension d (or zero module).

Includes: R, R_f, $H^i_I(R)$, $R_f(s)\cdot \underline{f}^s$ (over $D_{R(s)|\mathbb{C}(s)}$).

Consequences of B≥

B≥ (+ multiplicity)

$\Downarrow$

holonomic modules have finite length as D_R-modules

$\swarrow$ $\searrow$

every holonomic module (in particular, every local cohomology module) has finitely many associated primes as an R-module

every nonzero $f \in R$ has a Bernstein-Sato polynomial main theorem, vaguely

<u>Thm</u> (AHJNTW): $B\geq$ holds for certain graded singularities in a sense such that we can recover the consequences above.

II. Rings of differential operators

Def (Grothendieck): Given $A \subseteq R$ commutative rings, inductively define

$$D^0_{R|A} = \mathrm{Hom}_R(R,R) = \{\text{multiplication by } f \mid f \in R\}$$

$$D^i_{R|A} = \{\delta \in \mathrm{Hom}_A(R,R) \mid [\delta, f] \in D^{i-1}_{R|A} \text{ all } f \in R\}.$$

$$D_{R|A} = \bigcup_{i \in \mathbb{N}} D^i_{R|A} \subseteq \mathrm{Hom}_A(R,R)$$

is a (noncommutative) ring ~~that~~ naturally acts on R.

$D_{R|A}$ naturally acts on

R_f, $H^i_I(R)$ ...

examples of rings of differential operators

1) $S = \mathbb{C}[x_1, \dots, x_d]$

$$D_{S|\mathbb{C}} = S\langle \partial_1, \dots, \partial_d \rangle$$

(usual ring of diff'l operators)

* finitely gen $\mathbb{C}$-alg
* left & right Noetherian
* $gr^{ord}(D_{S|\mathbb{C}})$ is (commutative) poly ring

↳ structurally nice

* many operators of negative degree
* S is a simple $D_{S|\mathbb{C}}$-module

↳ action is potent

2) $T = \mathbb{C}[x_1, \dots, x_d]^{(2)} = \mathbb{C}[x_1^2, x_1x_2, \dots, x_d^2]$

[Kantor]

$$D_{T|\mathbb{C}} = D_{S|\mathbb{C}}^{(2)} = T\langle x_1\partial_1, x_1\partial_2, \dots, x_d\partial_d, \partial_1^2, \partial_1\partial_2, \dots, \partial_d^2 \rangle$$

- ✱ finitely gen $\mathbb{C}$-alg
- ✱ left & right Noetherian
- ✱ $\mathrm{gr}^{\mathrm{ord}}(D_{T|\mathbb{C}})$ is (commutative) f.g. $\mathbb{C}$-algebra

↪ structurally nice

- ✱ many operators of negative degree
- ✱ T is a simple $D_{T|\mathbb{C}}$-module

↪ action is potent

3) $\mathcal{U} = \mathbb{C}[x,y,z]/(x^3+y^3+z^3)$

[Bernstein-Gelfand-Gelfand]

$\mathcal{D}_{\mathcal{U}|\mathbb{C}}$

* NOT finitely gen $\mathbb{C}$-alg
* NOT 2-sided ideal Noetherian
* $gr^{ord}(\mathcal{D}_{\mathcal{U}|\mathbb{C}})$ is NON f.g. $\mathbb{C}$-algebra

↳ structurally bad

* NO operators of negative degree
* $\mathcal{U}$ is NOT a simple $\mathcal{D}_{\mathcal{U}|\mathbb{C}}$-module

↳ action is defective

$\mathbb{C} = \mathcal{U}/(x,y,z)$ is a $\mathcal{D}_{\mathcal{U}|\mathbb{C}}$-module of dimension zero $< 2 = \dim(\mathcal{U})$

$\Rightarrow$ no B≥ can hold on $\mathcal{U}$.

4) $V = \mathbb{C}[x,y,z,w] / (x^3+y^3+z^3+w^3)$

[Mallory]

$D_{V|\mathbb{C}}$

* NO operators of negative degree
* V is NOT a simple $D_{V|\mathbb{C}}$-module

1') $S' = \mathbb{F}_p[x_1, \dots, x_d]$

$$D_{S'/\mathbb{F}_p} = S'\langle \partial_1, \dots, \partial_d, \frac{1}{p!}\partial_1^p, \dots, \frac{1}{p!}\partial_d^p, \frac{1}{p^2!}\partial_1^{p^2}, \dots \rangle$$

$D_{S'/\mathbb{F}_p}$

- ✲ NOT finitely gen $\mathbb{F}_p$-alg
- ✲ NOT 2-sided ideal Noetherian
- ✲ $\mathrm{gr}^{ord}(D_{S'/\mathbb{F}_p})$ is NON f.g. $\mathbb{F}_p$-algebra

↳ structurally bad

- ✲ many operators of negative degree
- ✲ S' is a simple $D_{S'/\mathbb{F}_p}$-module

↳ action is potent

4') $V' = \mathbb{F}_p[x,y,z,w]/(x^3+y^3+z^3+w^3)$

$D_{V'/\mathbb{F}_p}$

* NOT f.g. $\mathbb{F}_p$-alg, NOT 2-sided noeth, grord$(D_{S'/\mathbb{F}_p})$ NOT f.g. $\mathbb{F}_p$-alg
* many ops of negative degree; V' simple$/D_{V'/\mathbb{F}_p}$ [Smith]

III. Filtrations & holonomicity

Recall: The Bernstein filtration on

$D_{\mathbb{C}[x_1,\dots,x_d]|\mathbb{C}}$ is

$$B^i = \bigoplus_{|a|+|b|\le i} \mathbb{C}\, x_1^{a_1}\cdots x_d^{a_d}\,\partial_1^{b_1}\cdots\partial_d^{b_d}$$

$$= \mathbb{C}\cdot\{\delta \text{ homogeneous} \mid \operatorname{ord}(\delta) + 2\deg(\delta) \le i\}$$

Can define this more generally:

Def: If R is a graded k-algebra, the Bernstein filtration with weight w on $D_{R|k}$ is $B_w^\bullet$, where

$$B_w^i = k\{\delta \text{ homogeneous} \mid \operatorname{ord}(\delta) + w\deg(\delta) \le i\}.$$

If R is gen. as a k-alg. in degree $< w$, then $\dim_k(B_w^i) < \infty$ all i

& $\bigcup_i B_w^i = D_{R|k}$.

A filtration $F^\bullet$ on a D_R-module M is compatible with $B_w^\bullet$ if

* $B_w^i \cdot F^j \subseteq F^{i+j}$
* each F^j is a fin dim k-vs.
* $\bigcup_j F^j = M$.

Say $(M, F^\bullet)$ is a $(D_R, B_w^\bullet)$-module for short.

dimension & multiplicity of a filtration

If $(M, F^\bullet)$ is a $(D_R, B^\bullet_W)$-module,

$$\dim(M, F^\bullet) = \inf\left\{ t \in \mathbb{R}_{\geq 0} \;\middle|\; \begin{array}{c} \dim_k F^i < i^t \\ \text{all } i \gg 0 \end{array} \right\}$$

$$e(M, F^\bullet) = \limsup_{i \to \infty} \frac{\dim_k F^i}{i^{\dim(M, F^\bullet)}}$$

(if $\dim(M, F^\bullet) < \infty$; ∞ otherwise)

If $R = \mathbb{C}[x_1, \ldots, x_d]$,

$$\underset{\text{(as defined earlier)}}{\dim(M)} = \inf\left\{ \dim(M, F^\bullet) \;\middle|\; \begin{array}{c} (M, F^\bullet) \text{ a} \\ (D_R, B^\bullet_2)\text{-module} \end{array} \right\}$$

Bavula showed B≥ for
polynomial rings of char p in the
sense of the definition above.

Lyubeznik gave characterization of
holonomic modules as
$\dim = d$ & finite multiplicity

IV. Bernstein algebras

Recall: A (noncommutative) algebra A is simple if

$$\forall a \in A \setminus 0, \quad 1 \in A \cdot a \cdot A.$$

Def: A (noncommutative) filtered algebra $(A, F^\bullet)$ is linearly simple if

$$\exists c : \forall a \in F^i \setminus 0, \quad 1 \in F^{ci} \cdot a \cdot F^{ci}.$$

Def: A (commutative) graded k-algebra R generated in degree $\leq v$ is a Bernstein algebra if for some ($\Leftrightarrow$ every) $w > v$,

* $(D_R, B_w^\bullet)$ is linearly simple
* $\dim(D_R, B_w^\bullet) = 2 \cdot \dim R$
* $e(D_R, B_w^\bullet) > 0$

<u>Th'm</u> (AHJNTW): If R is a Bernstein algebra, and $(M, F^\bullet)$ is a $(D_R, B^\bullet_W)$-module, then

B≥ holds / "holonomic" makes

1) $\dim(M, F^\bullet) \geq \dim R$

and if "=" holds, then

$0 < e(M, F^\bullet)$

"holonomic" modules have finite length

2) if $\dim(M, F^\bullet) = \dim R$ and $e(M, F^\bullet) < \infty$, then

M has finite length as a D_R-module.

the motivating D-modules are holonomic

3) $M = R, R_f, H^i_I(R)$,

and in char 0, $R_f(s) \cdot \underline{f^s}$ (over $D_{R(s)|k(s)}$) admit filtrations $F^\bullet$ with

$\dim(M, F^\bullet) = \dim R$ and $e(M, F^\bullet) < \infty$.

<u>proof of 1)</u> (cf. Joseph, Bavula)

Take c s.t. $\delta \in B_w^j \setminus 0 \Rightarrow 1 \in B^{cj} \cdot \delta \cdot B^{cj}$ for all j

and pick $\delta \in B_w^i \setminus 0$.

$$\delta B_w^{ci} F^i \subseteq \delta F^{(c+1)i}$$

$$\Rightarrow F^i \subseteq (B_w^{ci} \cdot \delta \cdot B_w^{ci}) \cdot F^i \subseteq B_w^{ci} \delta F^{(c+1)i}$$

So $F^i \neq 0 \Rightarrow \delta F^{(c+1)i} \neq 0$.

This means the action map

$$B_w^i \longrightarrow \mathrm{Hom}_k(F^{(c+1)i}, F^{(c+2)i})$$

is injective, so

$$\dim_k(B_w^i) \leq \dim_k(F^{(c+1)i}) \cdot \dim_k(F^{(c+2)i})$$

Then just a little analysis.

V. Classes of Bernstein algebras

<u>Thm</u> (AHJNTW): If

0) $R = K[x_1, \ldots, x_d]^G$ for $\mathrm{char}(K)=0$, G finite, or

p) R is a strongly F-regular graded K-algebra with finite F-representation type,

then R is a Bernstein algebra.

char p case includes

- invariants of finite groups
- toric rings
- coordinate ring of $Gr(2,n)$

Gives new proof of finiteness of $\mathrm{Ass}(H^i_I(R))$ in cases (0), (p) & of existence of BS polys in case (0).

Sketch of char (0) case

Want: $\delta \in B^t_{w,R^G} \setminus 0 \Rightarrow 1 \in B^{small}_{w,R^G} \cdot \delta \cdot B^{small}_{w,R^G}$
where "small" $\equiv$ linear in t

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reduce to case G has no "pseudoreflections"

Key point:
$$gr(B^0_{w,R^G}) \simeq gr(B^0_{w,R})^G$$
$\Rightarrow$ $B^0_{w,R}$ is a good filtration for D_R as a right (D_{R^G}, B^0_{w,R^G})-mod.

$\Rightarrow$ $\exists\, \gamma_1, \dots, \gamma_s \in D_R$: $B^j_{w,R} \subseteq \sum \gamma_i B^{small}_{w,R^G}$

WLOG, take $f \in D^{0}_{R^G} \left(= R^G\right) \cap B^{t}_{w,R}$

* $f^n \gamma_i \in B^{small}_{w,R} \cdot f$ for $n > \operatorname{ord}(\gamma_i)$

(small $\equiv$ linear in t)

* $$1 \in B^{small}_{w,R} \cdot f^n \cdot B^{small}_{w,R}$$

$$\subseteq \sum_i B^{small}_{w,R} \cdot f^n \gamma_i \cdot B^{small}_{w,R^G}$$

$$\subseteq \sum_i B^{small}_{w,R} \cdot f \cdot B^{small}_{w,R^G}$$

Apply Reynolds operator $\frac{1}{|G|} \sum_{g \in G} g$

$$\rightsquigarrow \quad 1 \in \sum_i B^{small}_{w,R^G} \cdot f \cdot B^{small}_{w,R^G} \qquad \square$$

char p case

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$$D_R = \bigcup_{e \in \mathbb{N}} \mathrm{End}_{R^{p^e}}(R)$$

"effective version of Smith–Van den Bergh"

VI. Nonexamples

* $\dfrac{\mathbb{C}[x,y,z,w]}{(x^3+y^3+z^3+w^3)}$ is NOT a Bernstein algebra:
  (though rational / KLT sings)
  $\mathbb{C}$ is a D-module [Mallory]

* $\dfrac{K[x,y]}{(xy)}$ is NOT a Bernstein algebra

  K is a D-module

* $\dfrac{K[r,s,t,u,v,w,x,y,z]}{(su^2x^2+sv^2y^2+tuvxy+rw^2z^2)}$ is NOT a Bernstein algebra
  (though rat'l / KLT OR str. F-reg sings)
  $H^3_{(x,y,z)}(R)$ w/ infinitely many associated primes [Singh Swanson]