Primary decomposition with differential operators

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Theorem (Palamodov, 1964)

Let $R = \mathbb{C}[x_1, \ldots, x_n]$, $\mathfrak{p} \subset R$ be a prime ideal and $Q \subset R$ be a $\mathfrak{p}$-primary ideal. Then, there exist **(Noetherian)** differential operators

$$A_1, \ldots, A_t \in D_n(\mathbb{C}) = R\langle \partial_{x_1}, \ldots, \partial_{x_n} \rangle,$$

such that

$$Q = \left\{ f \in R \mid A_j \cdot f \in \mathfrak{p} \text{ for all } 1 \leq j \leq t \right\}.$$

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Interesting for (at least) two reasons:

- Describe primary ideals via the use of differential operators.
- Fundamental Principle of Ehrenpreis and Palamodov: solutions of linear systems of PDE with constant coefficients.

A little bit of history and context

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⋆ *60's: The Fundamental Principle of Ehrenpreis and Palamodov states that the solutions of a linear system of PDE with constant coefficients can be represented in terms of certain integrals. At the core of the Fundamental Principal, one has **the existence of Noetherian operators for primary ideals in** $\mathbb{C}[x_1, \ldots, x_n]$.*

Palamodov's example

Let:

- $R = \mathbb{C}[x_1, x_2, x_3]$.
- $Q = \left(x_1x_3 - x_2, x_2^2, x_3^2\right) \subset R$.
- $\mathfrak{p} = (x_2, x_3) = \sqrt{I} \subset R$.

The Noetherian operators for I are $A_1 = 1$ and $A_2 = \partial_{x_3} + x_1\partial_{x_2}$, that is

$$Q = \left\{ f \in R \mid f \in \mathfrak{p} \;\text{ and }\; \left(\partial_{x_3} + x_1\partial_{x_2}\right) \cdot f \in \mathfrak{p} \right\}.$$

Objectives

1. Characterize primary ideals with the "use of differential operators" in more general classes of ring.
2. **Differential primary decompositions.**
3. Extension of Zariski-Nagata theorem.

Definition

Let R be a commutative ring and A be a subring. Let M, N be R-modules. The n-th order A-linear differential operators $\operatorname{Diff}^n_{R/A}(M, N) \subseteq \operatorname{Hom}_A(M, N)$ are defined inductively by:

1. $\operatorname{Diff}^0_{R/A}(M, N) := \operatorname{Hom}_R(M, N)$.
2. $\operatorname{Diff}^n_{R/A}(M, N) := \left\{\delta \in \operatorname{Hom}_A(M, N) \mid [\delta, r] \in \operatorname{Diff}^{n-1}_{R/A}(M, N) \text{ for all } r \in R\right\}$, where $[\delta, r](m) = \delta(rm) - r\delta(m)$ for all $m \in M$.

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Example

For $R = \mathbb{C}[x_1, \ldots, x_n]$ we have that

$$D_n(\mathbb{C}) = R\langle \partial_1, \ldots, \partial_n \rangle = \mathrm{Diff}_{R/\mathbb{C}}(R,R).$$

What "kind of differential operators" can we use to describe primary ideals?

BGG's example

Consider $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$, then Bernstein, Gelfand, Gelfand (1972) showed that:

1. $\mathrm{Diff}_{R/\mathbb{C}}(R, R)$ is not a Noetherian ring.
2. Let $\mathfrak{m} = (x, y, z) \subset R$. For all $\delta \in \mathrm{Diff}_{R/\mathbb{C}}(R, R)$, we have that $\delta(\mathfrak{m}) \subset \mathfrak{m}$.

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Observation

The differential operator "∂_x" does not exist in $\text{Diff}_{R/\mathbb{C}}(R, R)$ because we would obtain $0 = \partial_x(0) = \partial_x(x^3 + y^3 + z^3) = 3x^2$ (a contradiction).

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In $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$, $\mathfrak{m}^2$ cannot be described with differential operators

We **cannot** find $\delta_1, \ldots, \delta_m \in \mathrm{Diff}_{R/\mathbb{C}}(R,R)$ such that

$$\mathfrak{m}^2 = \left\{ f \in R \mid \delta_i \cdot f \in \mathfrak{m} \text{ for all } 1 \le i \le m \right\}.$$

Let $\mathbb{k}$ be a field, R be a $\mathbb{k}$-algebra of finite type, $\mathfrak{p} \in \mathrm{Spec}(R)$ be a prime ideal, and $Q \subset R$ be a $\mathfrak{p}$-primary ideal.

The canonical map $\pi : R \twoheadrightarrow R/\mathfrak{p}$ induces a map

$$\mathrm{Diff}_{R/\mathbb{k}}(\pi) : \mathrm{Diff}_{R/\mathbb{k}}(R, R) \to \mathrm{Diff}_{R/\mathbb{k}}(R, R/\mathfrak{p}), \qquad \delta \mapsto \overline{\delta} = \pi \circ \delta.$$

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If $\delta_1, \ldots, \delta_m \in \mathrm{Diff}_{R/\Bbbk}(R, R)$ and $Q = \left\{f \in R \mid \delta_i(f) \in \mathfrak{p} \text{ for all } 1 \leq i \leq m\right\}$ then

$$Q = \left\{f \in R \mid \overline{\delta_i}(f) = 0 \text{ for all } 1 \leq i \leq m\right\}.$$

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Set $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$ and $\mathfrak{m} = (x, y, z)$. In $\operatorname{Diff}_{R/\mathbb{C}}(R, R/\mathfrak{m})$ we do have the operator "$\overline{\partial_x}$" because

$$0 = \overline{\partial_x}(0) = \overline{\partial_x}(x^3 + y^3 + z^3) = \overline{3x^2} = 0.$$

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Therefore, $\mathrm{Diff}_{R/\Bbbk}(\pi)$ is **not surjective**.

Theorem (CR)

Let $\mathbb{k}$ be a field, R be a $\mathbb{k}$-algebra of finite type, $\mathfrak{p} \in \operatorname{Spec}(R)$ be a prime ideal, and $Q \subset R$ be a $\mathfrak{p}$-primary ideal. Then:

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2. If R is smooth over $\Bbbk$, then there exist $\delta_1, \ldots, \delta_m \in \mathrm{Diff}_{R/\Bbbk}(R, R)$ such that
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If R is smooth over $\Bbbk$, then $\operatorname{Diff}_{R/\Bbbk}(\pi)$ is surjective.

Let $\Delta_{R/\Bbbk} = \mathrm{Ker}(\mu)$ with $\mu : R \otimes_{\Bbbk} R \to R, r \otimes_{\Bbbk} s \mapsto rs$.

Proposition (Grothendieck 1967, Heyneman and Sweedler 1969)

We have the isomorphism

$$\mathrm{Hom}_R\left(P^n_{R/\Bbbk}(M), N\right) \xrightarrow{\cong} \mathrm{Diff}^n_{R/\Bbbk}(M,N), \quad \varphi \mapsto \varphi \circ d^n$$

where $d^n : M \to P^n_{R/\Bbbk}(M) := (R \otimes_{\Bbbk} M) \big/ \Delta^{n+1}_{R/\Bbbk}(R \otimes_{\Bbbk} M), \qquad w \mapsto \overline{1 \otimes_{\Bbbk} w}$.

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Proposition (Grothendieck 1967 EGA IV)

If R is smooth over $\Bbbk$, then the map

$$\text{Diff}_{R/\Bbbk}(\pi) : \text{Diff}_{R/\Bbbk}(R,R) \to \text{Diff}_{R/\Bbbk}(R, R/\mathfrak{p}), \qquad \delta \mapsto \overline{\delta} = \pi \circ \delta$$

is **surjective**.

Differential conditions for ideal membership

Data

$\Bbbk$ be a **perfect** field, R a $\Bbbk$-algebra of finite type, and $I \subset R$ an ideal with $\mathrm{Ass}(I) =: \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$.

Definition (CR, Sturmfels): Differential primary decomposition

A list of pairs $(\mathfrak{A}_1, \mathfrak{p}_1), \ldots, (\mathfrak{A}_k, \mathfrak{p}_k)$, where $\mathfrak{A}_i \subset \mathrm{Diff}_{R/\Bbbk}(R, R/\mathfrak{p}_i)$ is a finite subset, is a **differential primary decomposition for** I if for each $\mathfrak{p} \in \mathrm{Ass}(I)$:

$$I_\mathfrak{p} \;=\; \bigcap_{\substack{1 \le i \le k \\ \mathfrak{p}_i \subseteq \mathfrak{p}}} \left\{ f \in R_\mathfrak{p} \mid \delta'(f) = 0 \text{ for all } \delta \in \mathfrak{A}_i \right\}.$$

Here $\delta' \in \mathrm{Diff}_{R_\mathfrak{p}/\Bbbk}(R_\mathfrak{p}, R_\mathfrak{p}/\mathfrak{p}_i R_\mathfrak{p})$ denotes the localization of operator $\delta \in \mathfrak{A}_i$.

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- If $(\mathfrak{A}_1, \mathfrak{p}_1), \ldots, (\mathfrak{A}_k, \mathfrak{p}_k)$ is a **differential primary decomposition**, then
$$I = \left\{ f \in R \mid \delta(f) = 0 \text{ for all } \delta \in \mathfrak{A}_i \text{ and } 1 \le i \le k \right\}.$$
- If R is **smooth** over $\mathbb{k}$, then we can take $\mathfrak{A}_i \subset \operatorname{Diff}_{R/\mathbb{k}}(R, R)$.

• A differential primary decomposition always exists. Take a primary decomposition $I = Q_1 \cap \cdots \cap Q_k$ and set $\mathfrak{A}_i$ to be a set of Noetherian operators for the $\mathfrak{p}_i$-primary ideal Q_i.

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Example

Let $I = (x_1^3, x_1^2 x_2^2) \subset \Bbbk[x_1, x_2]$ with primary decomposition $I = (x_1^2) \cap (x_1^3, x_2^2)$.
We have Noetherian operators for the primary components:

$$(x_1^2) = \{f \in R \mid \delta(f) \in (x_1) \text{ for all } \delta \in \mathfrak{B}_1\},$$
$$(x_1^3, x_2^2) = \{f \in R \mid \delta(f) \in (x_1, x_2) \text{ for all } \delta \in \mathfrak{B}_2\},$$

where $\mathfrak{B}_1 = \{1, \partial_{x_1}\}$ and $\mathfrak{B}_2 = \{1, \partial_{x_1}, \partial_{x_2}, \partial_{x_1}\partial_{x_2}, \partial_{x_1}^2, \partial_{x_1}^2\partial_{x_2}\}$; this would yield a total of 8 differential operators.

But, we can describe I with just 6 differential operators:

$$I = \{f \in R \mid \delta(f) \in \mathfrak{p}_i \text{ for all } \delta \in \mathfrak{A}_i \text{ and } i = 1, 2\}$$

where $\mathfrak{A}_1 = \{1, \partial_{x_1}\}$ and $\mathfrak{A}_2 = \{\partial_{x_2}, \partial_{x_1}\partial_{x_2}, \partial_{x_1}^2, \partial_{x_1}^2\partial_{x_2}\}$.

Important question

What is the minimal size of a differential primary decomposition?

Defintion (**arithmetic multiplicity**)

$$\mathrm{amult}(I) := \mathrm{mult}_I(\mathfrak{p}_1) + \mathrm{mult}_I(\mathfrak{p}_2) + \cdots + \mathrm{mult}_I(\mathfrak{p}_k)$$

where

$$\mathrm{mult}_I(\mathfrak{p}_i) = \lambda_{R_{\mathfrak{p}_i}}\left(\mathrm{H}^0_{\mathfrak{p}_i}(R_{\mathfrak{p}_i}/IR_{\mathfrak{p}_i})\right).$$

Remark: the arithmetic multiplicity is **not** the sum of the multiplicities coming from a primary decomposition of I.

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Some connections

If I is homogeneous and $R = \mathbb{k}[x_1, \ldots, x_r]$, then the **arithmetic degree** is

$$\operatorname{adeg}(I) := \operatorname{mult}_I(\mathfrak{p}_1)\deg(\operatorname{Proj}(R/\mathfrak{p}_1)) + \cdots + \operatorname{mult}_I(\mathfrak{p}_k)\deg(\operatorname{Proj}(R/\mathfrak{p}_k)).$$

The arithmetic degree has been studied by: Bayer and Mumford (1993), Kollár (1988), Hartshorne (1966), Sturmfels, Trung, Vogel (1995).

Theorem (CR, Sturmfels)

Let $I \subset R$ and $\text{Ass}(I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$. The size of any differential primary decomposition is at least $\text{amult}(I)$ and this result is sharp. More precisely:

1. I has a differential primary decomposition with $|\mathfrak{A}_i| = \text{mult}_I(\mathfrak{p}_i)$.
2. A differential primary decomposition for I satisfies $|\mathfrak{A}_i| \geq \text{mult}_I(\mathfrak{p}_i)$.

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Slogan

The arithmetic multiplicity is the "differential complexity" of an ideal.

Remark

Our result also covers the case of modules.

Example

Let $R = \mathbb{k}[x,y,z]$ and $I = \left(x^2y, x^2z, xy^2, xyz^2\right)$ with primary decomposition

$$I = (x) \cap (x^2, y) \cap (y,z) \cap (x^2, y^2, z^2)$$

and associated primes $\mathfrak{p}_1 = (x), \mathfrak{p}_2 = (x,y), \mathfrak{p}_3 = (y,z), \mathfrak{p}_4 = (x,y,z)$. The Noetherian operators for the primary components are

$$(x) \;\Leftrightarrow\; \{1\}$$
$$(x^2, y) \;\Leftrightarrow\; \{1,\ \partial_x\}$$
$$(y,z) \;\Leftrightarrow\; \{1\}$$
$$(x^2, y^2, z^2) \;\Leftrightarrow\; \{1,\ \partial_x,\ \partial_y,\ \partial_z,\ \partial_x\partial_y,\ \partial_x\partial_z,\ \partial_y\partial_z,\ \partial_x\partial_y\partial_z\}.$$

This naive computation yields a differential primary decomposition of size 12.

However, a **minimal differential primary decomposition** for I is given by

$$\mathfrak{A}_1 = \{1\}, \mathfrak{A}_2 = \{\partial_x\}, \mathfrak{A}_3 = \{1\} \text{ and } \mathfrak{A}_4 = \{\partial_x\partial_y,\ \partial_x\partial_y\partial_z\}.$$

Notice that this coincides with $\text{amult}(I) = 5$. So, we have

$$I = \left\{f \in R \mid f \in \mathfrak{p}_1,\ f \in \mathfrak{p}_2,\ \tfrac{\partial f}{\partial x} \in \mathfrak{p}_3,\ \tfrac{\partial^2 f}{\partial x \partial y} \in \mathfrak{p}_4,\ \tfrac{\partial^3 f}{\partial x \partial y \partial z} \in \mathfrak{p}_4\right\}.$$

Example (continuation)

```
Macaulay2, version 1.17
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems, LLLBases, MinimalPrimes, Pr

i1 : load "modulesNoetherianOperators.m2"

i2 : R = QQ[x,y,z]

o2 = R

o2 : PolynomialRing

i3 : I = ideal(x^2*y,x^2*z,x*y^2,x*y*z^2)

               2    2      2       2
o3 = ideal (x y, x z, x*y , x*y*z )

o3 : Ideal of R

i4 : netList differentialPrimaryDecomposition I

     +--------------+--------------------+
o4 = |ideal x       |{| 1 |}             |
     +--------------+--------------------+
     |ideal (y, x)  |{| dx |}            |
     +--------------+--------------------+
     |ideal (z, y)  |{| 1 |}             |
     +--------------+--------------------+
     |ideal (z, y, x)|{| dxdy |, | dxdydz |}|
     +--------------+--------------------+
```

Zariski-Nagata Theorem

Let V be a variety in $\mathbb{C}^r$ and $\mathfrak{p} = I(V) \subset \mathbb{C}[x_1, \ldots, x_r]$. The n-th symbolic power of $\mathfrak{p}$ is defined as: $\mathfrak{p}^{(n)} = \mathfrak{p}^n R_{\mathfrak{p}} \cap R.$

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Easy reinterpretation (in the sense of Noetherian operators)

$\mathfrak{p}^{(n)} = \{f \in R \mid \frac{\partial^\alpha}{\partial x^\alpha}(f) \in \mathfrak{p} \quad \text{for all } |\alpha| \leq n-1\}$

Extension of Zariski-Nagata Theorem

Theorem (Dao, De Stefani, Grifo, Huneke, and Núñez-Betancourt 2018)

Let $\mathbb{k}$ be a perfect field, R be a **smooth** algebra of finite type over $\mathbb{k}$, and $\mathfrak{p} \in \operatorname{Spec}(R)$ be a prime ideal. Set

$$\mathfrak{p}^{\langle n \rangle} := \left\{ f \in R \mid \delta(f) \in \mathfrak{p} \text{ for all } \delta \in \operatorname{Diff}^{n-1}_{R/\mathbb{k}}(R,R) \right\}.$$

Then $\mathfrak{p}^{(n)} = \mathfrak{p}^{\langle n \rangle}$.

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We can proof the following further extension:

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Then:

- $\mathfrak{p}^{(n)} = \mathfrak{p}^{\{n\}}$.
- If R is **smooth** over $\Bbbk$, then $\mathfrak{p}^{(n)} = \mathfrak{p}^{\{n\}} = \mathfrak{p}^{\langle n \rangle}$.

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Thanks a lot!