Primary decomposition with differential operators

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Joint work with Bernd Sturmfels

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Theorem (Palamodov, 1964)

Let $R = \mathbb{C}[x_1, \ldots, x_n]$, $\mathfrak{p} \subset R$ be a prime ideal and $Q \subset R$ be a $\mathfrak{p}$-primary ideal. Then, there exist (Noetherian) differential operators

$$A_1, \ldots, A_t \in D_n(\mathbb{C}) = R\langle \partial_{x_1}, \ldots, \partial_{x_n} \rangle,$$

such that

$$Q = \{ f \in R \mid A_j \cdot f \in \mathfrak{p} \text{ for all } 1 \leq j \leq t \}.$$
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$A_1, \ldots, A_t$ are referred to as \textbf{Noetherian operators} that encode the $p$-primary ideal $Q$. 

Interesting for (at least) two reasons:

Describe primary ideals via the use of differential operators.

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- Describe primary ideals via the use of differential operators.
A little bit of history and context

- The existence of “Noetherian operators” for zero dimensional primary ideals is due to Gröbner in the 1930's.
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60’s: The Fundamental Principle of Ehrenpreis and Palamodov states that the solutions of a linear system of PDE with constant coefficients can be represented in terms of certain integrals. At the core of the Fundamental Principal, one has the existence of Noetherian operators for primary ideals in \( \mathbb{C}[x_1, \ldots, x_n] \).
Let:

- \( R = \mathbb{C}[x_1, x_2, x_3] \).
- \( Q = (x_1x_3 - x_2, x_2^2, x_3^2) \subset R \).
- \( \mathfrak{p} = (x_2, x_3) = \sqrt{I} \subset R \).

The Noetherian operators for \( I \) are \( A_1 = 1 \) and \( A_2 = \partial_x^3 + x_1 \partial_x^2 \), that is

\[
Q = \left\{ f \in R \mid f \in \mathfrak{p} \text{ and } (\partial_x^3 + x_1 \partial_x^2) \cdot f \in \mathfrak{p} \right\}.
\]
Objectives

1. Characterize primary ideals with the “use of differential operators” in more general classes of ring.

2. Differential primary decompositions.

Definition

Let $R$ be a commutative ring and $A$ be a subring. Let $M, N$ be $R$-modules. The $n$-th order $A$-linear differential operators $\text{Diff}^n_{R/A}(M, N) \subseteq \text{Hom}_A(M, N)$ are defined inductively by:

1. $\text{Diff}^0_{R/A}(M, N) := \text{Hom}_R(M, N)$.
2. $\text{Diff}^n_{R/A}(M, N) := \{ \delta \in \text{Hom}_A(M, N) \mid [\delta, r] \in \text{Diff}^{n-1}_{R/A}(M, N) \text{ for all } r \in R \}$, where $[\delta, r](m) = \delta(rm) - r\delta(m)$ for all $m \in M$.

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$$\text{Diff}_{R/A}(M, N) := \bigcup_{n=0}^{\infty} \text{Diff}^n_{R/A}(M, N).$$
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Example

For $R = \mathbb{C}[x_1, \ldots, x_n]$ we have that

$$D_n(\mathbb{C}) = R\langle \partial_1, \ldots, \partial_n \rangle = \text{Diff}_{R/\mathbb{C}}(R, R).$$
What “kind of differential operators” can we use to describe primary ideals?

Consider $R = \mathbb{C}[x, y, z] (x^3 + y^3 + z^3)$, then Bernstein, Gelfand, Gelfand (1972) showed that:

1. $\text{Diff}_{R/\mathbb{C}}(R, R)$ is not a Noetherian ring.

2. Let $m = (x, y, z) \subset R$. For all $\delta \in \text{Diff}_{R/\mathbb{C}}(R, R)$, we have that $\delta(m) \subset m$.

Observation: The differential operator "$\partial_x$" does not exist in $\text{Diff}_{R/\mathbb{C}}(R, R)$ because we would obtain $0 = \partial_x(0) = \partial_x(x^3 + y^3 + z^3) = 3x^2$ (a contradiction).

In $R = \mathbb{C}[x, y, z] (x^3 + y^3 + z^3)$, $m^2$ cannot be described with differential operators. We cannot find $\delta_1, \ldots, \delta_m \in \text{Diff}_{R/\mathbb{C}}(R, R)$ such that $m^2 = \{f \in R | \delta_i \cdot f \in m \text{ for all } 1 \leq i \leq m\}$.
What “kind of differential operators” can we use to describe primary ideals?

BGG’s example

Consider $R = \mathbb{C}[x, y, z] / (x^3 + y^3 + z^3)$, then Bernstein, Gelfand, Gelfand (1972) showed that:

1. $\operatorname{Diff}_{R/\mathbb{C}}(R, R)$ is not a Noetherian ring.

2. Let $m = (x, y, z) \subset R$. For all $\delta \in \operatorname{Diff}_{R/\mathbb{C}}(R, R)$, we have that $\delta(m) \subset m$. 
What “kind of differential operators” can we use to describe primary ideals?

**BGG’s example**

Consider \( R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)} \), then Bernstein, Gelfand, Gelfand (1972) showed that:

1. \( \text{Diff}_{\mathbb{C}}(R, R) \) is not a Noetherian ring.
2. Let \( m = (x, y, z) \subset R \). For all \( \delta \in \text{Diff}_{\mathbb{C}}(R, R) \), we have that \( \delta(m) \subset m \).

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The differential operator “\( \partial_x \)” does not exist in \( \text{Diff}_{\mathbb{C}}(R, R) \) because we would obtain \( 0 = \partial_x(0) = \partial_x(x^3 + y^3 + z^3) = 3x^2 \) (a contradiction).
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We cannot find \( \delta_1, \ldots, \delta_m \in \text{Diff}_{R/\mathbb{C}}(R, R) \) such that

\[ m^2 = \{ f \in R \mid \delta_i \cdot f \in m \text{ for all } 1 \leq i \leq m \}. \]
Let \( k \) be a field, \( R \) be a \( k \)-algebra of finite type, \( p \in \text{Spec}(R) \) be a prime ideal, and \( Q \subset R \) be a \( p \)-primary ideal.

The canonical map \( \pi : R \rightarrow R/p \) induces a map

\[
\text{Diff}_{R/k}(\pi) : \text{Diff}_{R/k}(R, R) \rightarrow \text{Diff}_{R/k}(R, R/p), \quad \delta \mapsto \overline{\delta} = \pi \circ \delta.
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Set $R = \mathbb{C}[x,y,z]_{(x^3+y^3+z^3)}$ and $m = (x, y, z)$. In $\text{Diff}_{R/\mathbb{C}}(R, R/m)$ we do have the operator "$\overline{\partial_x}$" because 

$$0 = \overline{\partial_x}(0) = \overline{\partial_x}(x^3 + y^3 + z^3) = 3x^2 = 0.$$
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$$0 = \overline{\partial_x}(0) = \overline{\partial_x}(x^3 + y^3 + z^3) = 3x^2 = 0.$$ 

Therefore, $\text{Diff}_{R/\mathbb{k}}(\pi)$ is not surjective.
Theorem (CR)

Let $\mathbb{k}$ be a field, $R$ be a $\mathbb{k}$-algebra of finite type, $\mathfrak{p} \in \text{Spec}(R)$ be a prime ideal, and $Q \subset R$ be a $\mathfrak{p}$-primary ideal. Then:

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2. If $R$ is smooth over $\mathbb{k}$, then there exist $\delta_1, \ldots, \delta_m \in \text{Diff}_{R/\mathbb{k}}(R, R)$ such that
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If $R$ is smooth over $\mathbb{k}$, then $\text{Diff}_{R/\mathbb{k}}(\pi)$ is surjective.
Let $\Delta_{R/\Bbbk} = \text{Ker}(\mu)$ with $\mu : R \otimes_{\Bbbk} R \to R$, $r \otimes_{\Bbbk} s \mapsto rs$.

**Proposition (Grothendieck 1967, Heyneman and Sweedler 1969)**

We have the isomorphism

$$\text{Hom}_R \left( P^n_{R/\Bbbk}(M), N \right) \cong \text{Diff}^n_{R/\Bbbk}(M, N), \quad \varphi \mapsto \varphi \circ d^n$$

where $d^n : M \to P^n_{R/\Bbbk}(M) := \left( R \otimes_{\Bbbk} M \right) / \Delta^{n+1}_{R/\Bbbk}(R \otimes_{\Bbbk} M)$, $w \mapsto 1 \otimes_{\Bbbk} w$. 
Let $\Delta_{R/k} = \text{Ker}(\mu)$ with $\mu : R \otimes_k R \to R, r \otimes_k s \mapsto rs$.

**Proposition (Grothendieck 1967, Heyneman and Sweedler 1969)**

We have the isomorphism

$$\text{Hom}_R \left( P^n_{R/k}(M), N \right) \overset{\cong}{\longrightarrow} \text{Diff}^n_{R/k}(M, N), \quad \varphi \mapsto \varphi \circ d^n$$

where $d^n : M \to P^n_{R/k}(M) := (R \otimes_k M) / \Delta^{n+1}_{R/k} (R \otimes_k M), \quad w \mapsto 1 \otimes_k w$.

**Proposition (Grothendieck 1967 EGA IV)**

If $R$ is smooth over $k$, then the map

$$\text{Diff}_{R/k}(\pi) : \text{Diff}_{R/k}(R, R) \to \text{Diff}_{R/k}(R, R/p), \quad \delta \mapsto \overline{\delta} = \pi \circ \delta$$

is surjective.
Differential conditions for ideal membership

Data

\( k \) be a perfect field, \( R \) a \( k \)-algebra of finite type, and \( I \subset R \) an ideal with \( \text{Ass}(I) =: \{ p_1, \ldots, p_k \} \).

Definition (CR, Sturmfels): Differential primary decomposition

A list of pairs \((A_1, p_1), \ldots, (A_k, p_k)\), where \( A_i \subset \text{Diff}_{R/k}(R, R/p_i) \) is a finite subset, is a differential primary decomposition for \( I \) if for each \( p \in \text{Ass}(I) \):

\[
I_p = \bigcap_{1 \leq i \leq k} \{ f \in R_p \mid \delta'(f) = 0 \text{ for all } \delta \in A_i \}.
\]

Here \( \delta' \in \text{Diff}_{R_p/k}(R_p, R_p/p_iR_p) \) denotes the localization of operator \( \delta \in A_i \).
Differential conditions for ideal membership

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\( \mathbb{k} \) be a **perfect** field, \( R \) a \( \mathbb{k} \)-algebra of finite type, and \( I \subset R \) an ideal with \( \text{Ass}(I) =: \{ p_1, \ldots, p_k \} \).

Definition (CR, Sturmfels): Differential primary decomposition

A list of pairs \( (\mathcal{A}_1, p_1), \ldots, (\mathcal{A}_k, p_k) \), where \( \mathcal{A}_i \subset \text{Diff}_{R/\mathbb{k}}(R, R/p_i) \) is a finite subset, is a **differential primary decomposition for** \( I \) if for each \( p \in \text{Ass}(I) \):

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Here \( \delta' \in \text{Diff}_{R_p/\mathbb{k}}(R_p, R_p/p_i R_p) \) denotes the localization of operator \( \delta \in \mathcal{A}_i \).

- If \( (\mathcal{A}_1, p_1), \ldots, (\mathcal{A}_k, p_k) \) is a **differential primary decomposition**, then
  \[
  I = \left\{ f \in R \mid \delta(f) = 0 \text{ for all } \delta \in \mathcal{A}_i \text{ and } 1 \leq i \leq k \right\}.
  \]
- If \( R \) is **smooth** over \( \mathbb{k} \), then we can take \( \mathcal{A}_i \subset \text{Diff}_{R/\mathbb{k}}(R, R) \).
A differential primary decomposition always exists. Take a primary decomposition $I = Q_1 \cap \cdots \cap Q_k$ and set $\mathcal{A}_i$ to be a set of Noetherian operators for the $p_i$-primary ideal $Q_i$.

Example

Let $I = (x_3^1, x_2^1 x_2^2) \subset k[x_1, x_2]$ with primary decomposition $I = (x_2^1) \cap (x_3^1, x_2^2)$.

We have Noetherian operators for the primary components:

- $(x_2^1) = \{ f \in R \mid \delta(f) \in (x_1) \text{ for all } \delta \in B_1 \}$,
- $(x_3^1, x_2^2) = \{ f \in R \mid \delta(f) \in (x_1, x_2) \text{ for all } \delta \in B_2 \}$,

where $B_1 = \{ 1, \partial x_1 \}$ and $B_2 = \{ 1, \partial x_1, \partial x_2, \partial x_1 \partial x_2, \partial x_1^2, \partial x_1 \partial x_2 \}$; this would yield a total of 8 differential operators.

But, we can describe $I$ with just 6 differential operators:

$I = \{ f \in R \mid \delta(f) \in p_i \text{ for all } \delta \in \mathcal{A}_i \}$

where $\mathcal{A}_1 = \{ 1, \partial x_1 \}$ and $\mathcal{A}_2 = \{ \partial x_2, \partial x_1 \partial x_2, \partial x_1^2, \partial x_1 \partial x_2 \}$.

Important question

What is the minimal size of a differential primary decomposition?
A differential primary decomposition always exists. Take a primary decomposition \( I = Q_1 \cap \cdots \cap Q_k \) and set \( A_i \) to be a set of Noetherian operators for the \( p_i \)-primary ideal \( Q_i \).

**Example**

Let \( I = (x_1^3, x_1^2 x_2^2) \subset k[x_1, x_2] \) with primary decomposition \( I = (x_1^2) \cap (x_1^3, x_2^2) \).

We have Noetherian operators for the primary components:

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(x_1^2) = \{ f \in R \mid \delta(f) \in (x_1) \text{ for all } \delta \in \mathcal{B}_1 \},
\]

\[
(x_1^3, x_2^2) = \{ f \in R \mid \delta(f) \in (x_1, x_2) \text{ for all } \delta \in \mathcal{B}_2 \},
\]

where \( \mathcal{B}_1 = \{ 1, \partial x_1 \} \) and \( \mathcal{B}_2 = \{ 1, \partial x_1, \partial x_2, \partial x_1 \partial x_2, \partial^2 x_1, \partial^2 x_1 \partial x_2 \} \); this would yield a total of 8 differential operators.

But, we can describe \( I \) with just 6 differential operators:

\[
I = \{ f \in R \mid \delta(f) \in p_i \text{ for all } \delta \in A_i \text{ and } i = 1, 2 \}
\]

where \( A_1 = \{ 1, \partial x_1 \} \) and \( A_2 = \{ \partial x_2, \partial x_1 \partial x_2, \partial^2 x_1, \partial^2 x_1 \partial x_2 \} \).

**Important question**

What is the minimal size of a differential primary decomposition?
**Definition (arithmetic multiplicity)**

\[ \text{amult}(I) := \text{mult}_I(p_1) + \text{mult}_I(p_2) + \cdots + \text{mult}_I(p_k) \]

where

\[ \text{mult}_I(p_i) = \lambda_{R_{p_i}} \left( H^0_{p_i} \left( R_{p_i} / IR_{p_i} \right) \right). \]

**Remark:** the arithmetic multiplicity is **not** the sum of the multiplicities coming from a primary decomposition of \( I \).
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**Some connections**

If \( I \) is homogeneous and \( R = \mathbb{k}[x_1, \ldots, x_r] \), then the **arithmetic degree** is

\[ \text{adeg}(I) := \text{mult}_I(p_1) \deg(\text{Proj}(R/p_1)) + \cdots + \text{mult}_I(p_k) \deg(\text{Proj}(R/p_k)). \]

The arithmetic degree has been studied by: Bayer and Mumford (1993), Kollár (1988), Hartshorne (1966), Sturmfels, Trung, Vogel (1995).
Theorem (CR, Sturmfels)

Let $I \subset R$ and $\text{Ass}(I) = \{p_1, \ldots, p_k\}$. The size of any differential primary decomposition is at least $\text{amult}(I)$ and this result is sharp. More precisely:

1. $I$ has a differential primary decomposition with $|\mathcal{A}_i| = \text{mult}_i(p_i)$.
2. A differential primary decomposition for $I$ satisfies $|\mathcal{A}_i| \geq \text{mult}_i(p_i)$. 

Slogan

The arithmetic multiplicity is the “differential complexity” of an ideal.

Remark

Our result also covers the case of modules.
Theorem (CR, Sturmfels)

Let $I \subset R$ and $\text{Ass}(I) = \{p_1, \ldots, p_k\}$. The size of any differential primary decomposition is at least $\text{mult}(I)$ and this result is sharp. More precisely:

1. $I$ has a differential primary decomposition with $|A_i| = \text{mult}_I(p_i)$.
2. A differential primary decomposition for $I$ satisfies $|A_i| \geq \text{mult}_I(p_i)$.

Slogan

The arithmetic multiplicity is the “differential complexity” of an ideal.

Remark

Our result also covers the case of modules.
Example

Let $R = \mathbb{k}[x, y, z]$ and $I = (x^2 y, x^2 z, xy^2, xyz^2)$ with primary decomposition

$$I = (x) \cap (x^2, y) \cap (y, z) \cap (x^2, y^2, z^2)$$

and associated primes $p_1 = (x), p_2 = (x, y), p_3 = (y, z), p_4 = (x, y, z)$. The Noetherian operators for the primary components are

$$
(x) \iff \{1\}
$$

$$
(x^2, y) \iff \{1, \partial_x\}
$$

$$
(y, z) \iff \{1\}
$$

$$
(x^2, y^2, z^2) \iff \{1, \partial_x, \partial_y, \partial_z, \partial_x\partial_y, \partial_x\partial_z, \partial_y\partial_z, \partial_x\partial_y\partial_z\}.
$$

This naive computation yields a differential primary decomposition of size 12.

However, a **minimal differential primary decomposition** for $I$ is given by

$$
\mathcal{A}_1 = \{1\}, \mathcal{A}_2 = \{\partial_x\}, \mathcal{A}_3 = \{1\} \text{ and } \mathcal{A}_4 = \{\partial_x\partial_y, \partial_x\partial_y\partial_z\}.
$$

Notice that this coincides with $\text{amult}(I) = 5$. So, we have

$$
I = \{f \in R \mid f \in p_1, f \in p_2, \frac{\partial f}{\partial x} \in p_3, \frac{\partial^2 f}{\partial x\partial y} \in p_4, \frac{\partial^3 f}{\partial x\partial y\partial z} \in p_4\}.
$$
Example (continuation)

Macaulay2, version 1.17
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems, LLLBases, MinimalPrimes, PrimaryDecomposition

i1 : load "modulesNoetherianOperators.m2"

i2 : R = QQ[x,y,z]
o2 = R

o2 : PolynomialRing

i3 : I = ideal(x^2*y,x^2*z,x*y^2,x*y*z^2)

 2 2 2 2
o3 = ideal (x y, x z, x y , x y z )

o3 : Ideal of R

i4 : netList differentialPrimaryDecomposition I

+-------------------------------------------------+
| ideal x |{1 1} |
+--------------------------+
| ideal (y, x) |{1 dx} |
+--------------------------+
| ideal (z, y) |{1 1} |
+--------------------------+
| ideal (z, y, x) |{1 dxdy} |{1 dxdydz} |
+--------------------------+
Zariski-Nagata Theorem

Let $V$ be a variety in $\mathbb{C}^r$ and $p = I(V) \subset \mathbb{C}[x_1, \ldots, x_r]$. The $n$-th symbolic power of $p$ is defined as: $p^{(n)} = p^n R_p \cap R$. 

Easy reinterpretation (in the sense of Noetherian operators) $p^{(n)} = \{ f \in R | \partial^\alpha f \in p \text{ for all } |\alpha| \leq n-1 \}$. 

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Primary decomposition with differential operators
Zariski-Nagata Theorem

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Zariski-Nagata Theorem

For all $n \geq 1$ we have

- $p^{(n)} = \bigcap_{q \in V} m_q^n$. ($m_q$ is the maximal ideal corresponding with $q$)
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Zariski-Nagata Theorem

For all $n \geq 1$ we have

- $p^{(n)} = \bigcap_{q \in V} m_q^n$. (where $m_q$ is the maximal ideal corresponding with $q$)
- $p^{(n)} = \{ f \in R \mid \frac{\partial^\alpha f}{\partial x^\alpha} (q) = 0$ for all $|\alpha| \leq n - 1$ and $q \in V \}$.
Let $V$ be a variety in $\mathbb{C}^r$ and $\mathfrak{p} = I(V) \subset \mathbb{C}[x_1, \ldots, x_r]$. The $n$-th symbolic power of $\mathfrak{p}$ is defined as: $\mathfrak{p}^{(n)} = \mathfrak{p}^n R_\mathfrak{p} \cap R$.

Zariski-Nagata Theorem

For all $n \geq 1$ we have

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Let $V$ be a variety in $\mathbb{C}^r$ and $\mathfrak{p} = I(V) \subset \mathbb{C}[x_1, \ldots, x_r]$. The $n$-th symbolic power of $\mathfrak{p}$ is defined as: $\mathfrak{p}^{(n)} = \mathfrak{p}^n R_{\mathfrak{p}} \cap R$.

**Zariski-Nagata Theorem**

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**Easy reinterpretation (in the sense of Noetherian operators)**

$\mathfrak{p}^{(n)} = \{ f \in R \mid \frac{\partial^\alpha f}{\partial x^\alpha} (f) \in \mathfrak{p} \text{ for all } |\alpha| \leq n - 1 \}$
Theorem (Dao, De Stefani, Grifo, Huneke, and Núñez-Betancourt 2018)

Let $k$ be a perfect field, $R$ be a smooth algebra of finite type over $k$, and $p \in \text{Spec}(R)$ be a prime ideal. Set

$$p^{(n)} := \{ f \in R \mid \delta(f) \in p \text{ for all } \delta \in \text{Diff}_{R/k}^{n-1}(R, R) \}.$$

Then $p^{(n)} = p^{(n)}$. 

We can prove the following further extension:

Theorem (CR)

Let $k$ be a perfect field, $R$ be an algebra of finite type over $k$, and $p \in \text{Spec}(R)$ be a prime ideal. Set

$$p^{\{n\}} := \{ f \in R \mid \delta(f) = 0 \text{ for all } \delta \in \text{Diff}_{R/k}^{n-1}(R, R) \}.$$

Then $p^{(n)} = p^{\{n\}}$. If $R$ is smooth over $k$, then $p^{(n)} = p^{\{n\}} = p^{(n)}$. 

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Primary decomposition with differential operators
Extension of Zariski-Nagata Theorem

Theorem (Dao, De Stefani, Grifo, Huneke, and Núñez-Betancourt 2018)

Let $k$ be a perfect field, $R$ be a smooth algebra of finite type over $k$, and $p \in \text{Spec}(R)$ be a prime ideal. Set

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Then $p^{(n)} = p^{\langle n \rangle}$.

We can prove the following further extension:

Theorem (CR)

$k$ perfect field, $R$ algebra of finite type over $k$, and $p \in \text{Spec}(R)$. Set

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Then:

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Primary decomposition with differential operators
Extension of Zariski-Nagata Theorem

**Theorem (Dao, De Stefani, Grifo, Huneke, and Núñez-Betancourt 2018)**

Let $\mathbb{k}$ be a perfect field, $R$ be a smooth algebra of finite type over $\mathbb{k}$, and $p \in \text{Spec}(R)$ be a prime ideal. Set

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Then $p^{(n)} = p^{(n)}$.

We can proof the following further extension:

**Theorem (CR)**

Let $\mathbb{k}$ be a perfect field, $R$ algebra of finite type over $\mathbb{k}$, and $p \in \text{Spec}(R)$. Set

$$p\{n\} := \{ f \in R \mid \delta(f) = 0 \text{ for all } \delta \in \text{Diff}^{n-1}_{R/\mathbb{k}}(R, R/p) \}.$$

Then:

- $p^{(n)} = p\{n\}$.
- If $R$ is smooth over $\mathbb{k}$, then $p^{(n)} = p\{n\} = p^{(n)}$. 

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Thanks a lot!