Pure subrings of polynomial rings.

A ring homomorphism $R \to S$ is <u>pure</u> if $(R \to S) \otimes_R M$ is injective for each R-module M.

- We say R is a <u>pure</u> <u>subring</u> of S.

- Faithfully flat $\Longrightarrow$ pure

  $\not\Leftarrow$ For example, $k[x^2, xy, y^2] \subseteq k[x,y]$ is pure

- If $R \hookrightarrow S$ is R-split, then it is pure.

  $\Leftarrow$ holds if R is complete local or if R, S are $\mathbb{N}$-graded with $R_0 = S_0$ a field.

- $k[x] \hookrightarrow \dfrac{k[x,y]}{(xy^2 - y + 1)}$ is pure but not $k[x]$-split [CGM].

<u>Question</u> : Let R be a f.g. algebra over A (field / $\mathbb{Z}$ / $\hat{\mathbb{Z}}_{(p)}$).
Is R a pure subring of a polynomial ring over A?

- $R := \dfrac{\mathbb{Q}[x,y,z]}{(x^2+y^2+z^2)}$ is not unirational, though it is a pure subring of $\mathbb{C}[s,t]$.

$$R \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathbb{C}[s^2, st, t^2] \subseteq \mathbb{C}[s,t].$$

- A pure subring of a normal ring is normal so $\mathbb{C}[x^2, x^3]$ is not a pure subring of a pol ring.

- Hochster : A normal affine semigroup ring is a pure subring of a polynomial ring (in new variables)

Let K be a field and G a linearly reductive subgp of $GL_n(K)$ acting on a polynomial ring S over K. Then S^G is a pure subring of S. In particular, the following are pure subrings of polynomial rings:

- Determinantal rings

$$\mathbb{C}[X]/I_t(X) \cong \mathbb{C}[YZ] \hookrightarrow \mathbb{C}[Y,Z]$$

$X : r \times s$

$Y : r \times t-1$

$Z : t-1 \times s$

- Symmetric determinantal rings

$$\mathbb{C}[X]/I_t(X) \cong \mathbb{C}[Y^{tr} Y] \hookrightarrow \mathbb{C}[Y]$$

$X : n \times n$ sym.

$Y : t-1 \times n$

- Pfaffian determinantal rings

$$\mathbb{C}[X]/Pf_{2t}(X) \cong \mathbb{C}[Y^{tr}\Omega Y] \hookrightarrow \mathbb{C}[Y]$$

$X : n \times n$ alt

$Y : 2t-2 \times n$

$\Omega : \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

Let R be an $\mathbb{N}$-graded ring, fin. gen. over a field $K := R_0$.
Suppose R is a pure subring of a polynomial ring. Then:

- R is normal, Cohen-Macaulay (Hochster-Roberts)
- R has rat'l singularities if $\operatorname{char} K = 0$ (Boutot)
- R is F-regular if $\operatorname{char} K > 0$ (Hochster-Huneke)
- $a(R) < 0$ provided $\dim R > 0$.

$R := k[x_1, \ldots, x_n]/(f)$ has $a(R) = \deg f - n$

If $\deg f \geq n$, R is not a pure subring of a pol. ring

<u>Shioda</u>: $\overline{\mathbb{F}_p}[w,x,y,z]/(w^{p+1}+x^{p+1}+y^{p+1}+z^{p+1})$ is unirational $(p \geq 3)$

Núñez-Betancourt: If $R \hookrightarrow S$ is pure, $\mathfrak{a}$ an ideal of R, then

$$\operatorname{Ass} H^k_{\mathfrak{a}}(S) \text{ finite} \implies \operatorname{Ass} H^k_{\mathfrak{a}}(R) \text{ finite}$$

Huneke-Sharp/Lyubeznik: If S is a pol. ring over a field, then $\operatorname{Ass} H^k_{\mathfrak{a}}(S)$ is finite

S.-Swanson: K a field, $R := \dfrac{K[r,s,t,u,v,w,x,y,z]}{(su^2x^2 + sv^2y^2 + tuvxy + rw^2z^2)}$

has rat'l singularities / is F-regular, but

$$\operatorname{Ass} H^3_{(x,y,z)}(R) \text{ is infinite.}$$

R is not a pure subring of a polynomial ring

Smith: Suppose $R \hookrightarrow S$ is R-split where, say, R, S are K-algebras. Then

S is $D_{S|K}$ simple $\Longrightarrow$ R is $D_{R|K}$ simple

Mallory: $R := \dfrac{\mathbb{C}[w,x,y,z]}{(w^3+x^3+y^3+z^3)}$. Then $\left[D_{R|\mathbb{C}}\right]_{<0} = 0$.

Hence $R \supsetneq R_{\geq 1} \supsetneq R_{\geq 2} \supsetneq \cdots$ are $D_{R|\mathbb{C}}$-modules.

So R is not a direct summand of any pol ring $/\mathbb{C}$.

Question: Set $R := \dfrac{\mathbb{F}_p[w,x,y,z]}{(w^3+x^3+y^3+z^3)}$ where $p \geq 5$.

Is R a direct summand of a pol ring?

<u>Question</u>: Is $R := \mathbb{F}_p[u,v,w,x,y,z]/(ux - vy + wz)$ direct summand of a polynomial ring?

<u>Note</u>: $ux - vy + wz = \operatorname{pfaff}\begin{pmatrix} 0 & u & v & w \\ -u & 0 & z & y \\ -v & -z & 0 & x \\ -w & -y & -x & 0 \end{pmatrix}$

Jeffries, S. : Let A be $\mathbb{Z}$ or $\hat{\mathbb{Z}}_{(p)}$ and R be any of :

① $A[X]/\det X$ $\quad X : n \times n$ where $n \geq 3$

② $A[X]/\mathrm{pfaff}\, X$ $\quad X : n \times n$ alt, $n \geq 4$ even

③ $A[X]/\det X$ $\quad X : n \times n$ sym, $\begin{cases} n \geq 4 \\ \text{OR} \\ n = 3,\ p = 2 \end{cases}$

Then R is <u>not</u> a direct summand of a polynom-ring $/ A$.

<u>Idea</u>: Frobenius lifting ; work of Zdanowicz

Suppose p is a prime integer that is not a unit in a ring S. A lift of the Frobenius endomorphism F of S/pS is an endomorphism Λ_p of S such that

$$\begin{tikzcd} S \arrow[r, "\Lambda_p"] \arrow[d] & S \arrow[d] \\ S/pS \arrow[r, "F"] & S/pS \end{tikzcd}$$

commutes.

If $R \hookrightarrow S$ is R-split then

S/p^2S has a Frob lift $\Longrightarrow$ R/p^2R has a Frob. lift.

3x3 symmetric matrix of indeterminates / det

$$\det \begin{pmatrix} 2ux & uy+vx & uz+wx \\ uy+vx & 2vy & vz+wy \\ uz+wx & vz+wy & 2wz \end{pmatrix} = 0$$

If X is a 3x3 sym. matrix,

$$\frac{\mathbb{Z}[x]}{\det x}\left[\tfrac{1}{2}\right] \simeq \mathbb{Z}_2[ux,\ vy,\ wz,\ uy+vx,\ uz+wx,\ vz+wy]$$

$$= \mathbb{Z}_2[ux, uy, uz, vx, vy, vz, wx, wy, wz]^G$$

where G is generated by the involution induced by

$u \longleftrightarrow x$, $v \longleftrightarrow y$, $w \longleftrightarrow z$