The Hodge filtration on local cohomology

Mircea Mustață
(joint work with Mihnea Popa)

ICERM (virtual)
August 9, 2021

The setup

$X = \mathrm{Spec}(R)$ smooth complex affine algebraic variety, of dimension n, and $Z = V(I) \subseteq X$ a closed subset. We put $U = X \smallsetminus Z$ and $j\colon U \hookrightarrow X$ is the inclusion.

We have the *local cohomology modules* $H^q_I(R)$ for $q \geq 0$. As it is well-known, part of the difficulty in studying these comes from the fact that they are not finitely generated R-modules.

The setup

$X = \mathrm{Spec}(R)$ smooth complex affine algebraic variety, of dimension n, and $Z = V(I) \subseteq X$ a closed subset. We put $U = X \smallsetminus Z$ and $j \colon U \hookrightarrow X$ is the inclusion.

We have the *local cohomology modules* $H^q_I(R)$ for $q \geq 0$. As it is well-known, part of the difficulty in studying these comes from the fact that they are not finitely generated R-modules.

However, they are *finitely generated* (even *holonomic*) as modules over the ring D_R of differential operators on R. Lyubeznik (1993) exploited this structure to prove various finiteness statements for the local cohomology modules.

The setup, cont'd

The D_R-module structure follows easily from the computation via the Čech complex: if $I = (f_1, \ldots, f_r)$, then $H_I^q(R)$ is the q^{th} cohomology of the complex

$$0 \to R \to \bigoplus_i R_{f_i} \to \bigoplus_{i<j} R_{f_i f_j} \to \ldots \to R_{f_1 \cdots f_r} \to 0.$$

Note that each $R_{f_{i_1} \cdots f_{i_k}}$ is naturally a D_R-module and the maps in the complex are D_R-linear, hence the cohomology carries a D_R-module structure.

A trivial example: if $I = (f)$ is principal, then $H_I^q(R) = 0$ for $q \neq 1$ and

$$H_I^1(R) = R_f / R.$$

Local cohomology as mixed Hodge modules

In fact, the D_R-modules $H^q_I(M)$ have more structure: they are *mixed Hodge modules* in the sense of Saito's theory. For the purpose of this talk, this means that they carry a canonical good filtration, the *Hodge filtration* (they also carry a *weight filtration*, but we will not be concerned with this).

Local cohomology as mixed Hodge modules

In fact, the D_R-modules $H_I^q(M)$ have more structure: they are *mixed Hodge modules* in the sense of Saito's theory. For the purpose of this talk, this means that they carry a canonical good filtration, the *Hodge filtration* (they also carry a *weight filtration*, but we will not be concerned with this). Recall: D_R carries the filtration by order of differential operators. If $x_1, \ldots, x_n$ are algebraic coordinates on X, then

$$F_p D_R = \bigoplus_{\alpha_1+\ldots+\alpha_n \leq p} R\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \subseteq \mathrm{End}_{\mathbf{C}}(R).$$

The filtration condition for $\big(H_I^q(M), F_\bullet\big)$ is that

$$F_p D_R \cdot F_i H_I^q(M) \subseteq F_{p+i} H_I^q(M) \quad \text{for all} \quad p, i \in \mathbf{Z}$$

It is a *good filtration* if there is i_0 such that the above is an equality for $i = i_0$ and all $p \geq 0$ (one says the filtration is *generated at level* i_0).

Local cohomology as mixed Hodge modules, cont'd

In previous work, we studied the case when $I = (f)$ is principal, when understanding the Hodge filtration on $H^1_I(R)$ is equivalent to understanding the Hodge filtration of R_f.

In this case, one can show that for $p \geq 0$:

$$F_p R_f \subseteq \frac{1}{f^{p+1}} R.$$

The difference is measured by the *Hodge ideals* $I_p(f)$:

$$F_p R_f = I_p(f) \cdot \frac{1}{f^{p+1}} R$$

Local cohomology as mixed Hodge modules, cont'd

In previous work, we studied the case when $I = (f)$ is principal, when understanding the Hodge filtration on $H^1_I(R)$ is equivalent to understanding the Hodge filtration of R_f.

In this case, one can show that for $p \geq 0$:

$$F_p R_f \subseteq \frac{1}{f^{p+1}} R.$$

The difference is measured by the *Hodge ideals* $I_p(f)$:

$$F_p R_f = I_p(f) \cdot \frac{1}{f^{p+1}} R$$

For arbitrary I, given a system of generators, the maps in the Čech complex are strict with respect to the Hodge filtration and the induced filtration on cohomology is the Hodge filtration on the local cohomology modules.

An example: the case of smooth subvarieties

Let $R = \mathbf{C}[x_1, \ldots, x_n]$ and $I = (x_1, \ldots, x_r)$. The only nonzero local cohomology is

$$H^r_I(R) = \mathrm{coker}\big(\oplus_i R_{x_1 \cdots \widehat{x_i} \cdots x_n} \to R_{x_1 \cdots x_n} \big)$$

and $F_p H^r_I(R)$ is generated over R by $\frac{1}{x_1^{k_1} \cdots x_r^{k_r}}$, with $k_i \geq 1$ for all i and $k_1 + \ldots + k_r = p + r$.

Note: in this case the Hodge filtration is generated at level 0.

An example: the case of smooth subvarieties

Let $R = \mathbf{C}[x_1, \ldots, x_n]$ and $I = (x_1, \ldots, x_r)$. The only nonzero local cohomology is

$$H_I^r(R) = \mathrm{coker}\big(\oplus_i R_{x_1 \cdots \widehat{x_i} \cdots x_n} \to R_{x_1 \cdots x_n} \big)$$

and $F_p H_I^r(R)$ is generated over R by $\frac{1}{x_1^{k_1} \cdots x_r^{k_r}}$, with $k_i \geq 1$ for all i and $k_1 + \ldots + k_r = p + r$.

Note: in this case the Hodge filtration is generated at level 0.

Very few other examples are known (beyond codimension 1). The most interesting one: that of a generic determinantal variety in $\mathbf{C}^{m \times n}$ (Perlman-Raicu and Perlman).

Computation via log resolutions

Suppose that $g\colon Y \to X$ is a log resolution of (X, Z), that is

- g is projective
- g is an isomorphism over $U = X \smallsetminus Z$
- Y is smooth
- $g^{-1}(Z)_{\rm red}$ is a simple normal crossing divisor E.

Computation via log resolutions

Suppose that $g\colon Y \to X$ is a log resolution of (X, Z), that is

- g is projective
- g is an isomorphism over $U = X \smallsetminus Z$
- Y is smooth
- $g^{-1}(Z)_{\mathrm{red}}$ is a simple normal crossing divisor E.

For $q \geq 2$, if $j\colon U \hookrightarrow X$ is the inclusion, we have

$$H^q_I(R) = R^{q-1} j_* \mathcal{O}_U$$

and we also have a short exact sequence

$$0 \to \mathcal{O}_X \to j_*\mathcal{O}_U \to H^1_I(R) \to 0.$$

We thus need to compute the Hodge filtration on $R^{q-1}j_*\mathcal{O}_U$, or on the corresponding right D_R-module $R^{q-1}j_*\omega_U$.

Computation via log resolutions, cont'd

The commutative diagram

$$\begin{array}{ccc} g^{-1}(U) & \xrightarrow{j'} & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ U & \xrightarrow{j} & X \end{array}$$

gives $R^{q-1}j_*\mathcal{O}_U \simeq R^{q-1}g_*\mathcal{O}_Y(*E)$, where $\mathcal{O}_Y(*E) = j'_*\mathcal{O}_{g^{-1}(U)}$.

This is useful since

• The right $\mathcal{D}_Y$-modules $\omega_Y(*E)$ corresponding to $\mathcal{O}_Y(*E)$ has an explicit filtered resolution by a log de Rham complex.

• Push-forward via the projective morphism g is well-behaved (Saito's Strrictness theorem).

Computation via log resolutions, cont'd

More precisely, $\omega_Y(*E)$ is resolved by the filtered complex $A^\bullet$:

$$0 \to \mathcal{D}_Y \to \Omega^1_Y(\log E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \to \ldots \to \Omega^n_Y(\log E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \to 0$$

Saito's Strictness theorem then gives

$$F_p R^{q-1} j_* \mathcal{O}_U \otimes_R \omega_R \simeq R^{q-1} g_* F_{p-n}(A^\bullet \otimes_{\mathcal{D}_Y} g^* \mathcal{D}_X)$$

where $F_{p-n}(A^\bullet \otimes_{\mathcal{D}_Y} g^* \mathcal{D}_X)$ is the complex

$$0 \to g^* F_{p-n} \mathcal{D}_X \to \Omega^1_Y(\log E) \otimes_{\mathcal{O}_Y} g^* F_{p-n+1} \mathcal{D}_X \to \ldots$$

$$\ldots \to \Omega^n_Y(\log E) \otimes_{\mathcal{O}_Y} g^* F_p \mathcal{D}_X \to 0$$

Consequences of the birational description

1) Description of the first piece of the Hodge filtration:

$$F_0 H_I^q(R) \otimes_R \omega_R = R^{q-1} g_* \omega_E \quad \text{for all} \quad q$$

Consequences of the birational description

1) Description of the first piece of the Hodge filtration:

$$F_0 H_I^q(R) \otimes_R \omega_R = R^{q-1} g_* \omega_E \quad \text{for all} \quad q$$

We have a canonical factorization

$$R^{q-1} g_* \omega_E \xrightarrow{\alpha_q} \mathrm{Ext}_R^q(R/I, \omega_R) \xrightarrow{\beta_q} H_I^q(\omega_R)$$

where α_q is obtained from $\mathcal{O}_Z \to \mathbf{R} g_* \mathcal{O}_E$ applying $\mathbf{R}\mathrm{H}om_{R/I}(-, \omega_{R/I}^\bullet)$ and taking cohomology.

In particular, it follows from this picture that all α_q are injective (result of Kovács-Schwede, useful when studying Du Bois singularities).

Consequences of the birational description, cont'd

2) We get a necessary and sufficient condition for the Hodge filtration on $H_I^q(R)$ to be generated at level k when the higher local cohomology vanishes: $H_I^j(R) = 0$ for $j > q$.
Note now that $H_I^q(R) = 0$ if and only if the Hodge filtration is generated at level 0 and $F_0 H_I^q(R) = 0$. We thus obtain the following description of the *local cohomological dimension* of Z in X:

Consequences of the birational description, cont'd

2) We get a necessary and sufficient condition for the Hodge filtration on $H^q_I(R)$ to be generated at level k when the higher local cohomology vanishes: $H^j_I(R) = 0$ for $j > q$.
Note now that $H^q_I(R) = 0$ if and only if the Hodge filtration is generated at level 0 and $F_0 H^q_I(R) = 0$. We thus obtain the following description of the *local cohomological dimension* of Z in X:

Theorem 1 (M-, Popa). For a positive integer c, the following are equiv:

i) $H^q_I(R) = 0$ for all $q > c$.

ii) $R^{j+i} g_* \Omega^{n-i}_Y(\log E) = 0$ for $j \geq c$ and $i \geq 0$.

Consequences of the birational description, cont'd

2) We get a necessary and sufficient condition for the Hodge filtration on $H^q_I(R)$ to be generated at level k when the higher local cohomology vanishes: $H^j_I(R) = 0$ for $j > q$.
Note now that $H^q_I(R) = 0$ if and only if the Hodge filtration is generated at level 0 and $F_0 H^q_I(R) = 0$. We thus obtain the following description of the *local cohomological dimension* of Z in X:

Theorem 1 (M-, Popa). For a positive integer c, the following are equiv:

i) $H^q_I(R) = 0$ for all $q > c$.

ii) $R^{j+i} g_* \Omega^{n-i}_Y(\log E) = 0$ for $j \geq c$ and $i \geq 0$.

For example, by taking $c = n - 1$ and $c = n - 2$, we recover (with some worrk) old results of Hartshorne-Lichtenbaum and Ogus. It would be interesting to relate this to an important description of local cohomological dimension via local de Rham cohomology, due to Ogus.

Comparison with other filtrations

Since $\left(H^q_I(R), F_\bullet\right)$ underlies a mixed Hodge module supported on Z:

$$I \cdot F_p H^q_I(R) \subseteq F_{p-1} H^q_I(R) \quad \text{for all} \quad p.$$

We have also seen that $I \cdot F_0 H^q_I(R) = 0$, hence

$$F_p H^q_I(R) \subseteq O_p H^q_I(R) := \{u \in H^q_I(R) \mid I^{p+1} u = 0\}$$

Comparison with other filtrations

Since $\big(H^q_I(R), F_\bullet\big)$ underlies a mixed Hodge module supported on Z:

$$I \cdot F_p H^q_I(R) \subseteq F_{p-1} H^q_I(R) \quad \text{for all} \quad p.$$

We have also seen that $I \cdot F_0 H^q_I(R) = 0$, hence

$$F_p H^q_I(R) \subseteq O_p H^q_I(R) := \{u \in H^q_I(R) \mid I^{p+1} u = 0\}$$

If $q = \mathrm{codim}(I)$, then

$$O_p H^q_I(R) = E_p H^q_I(R) := \mathrm{Im}\big(\mathrm{Ext}^q_R(R/I^{p+1}, R) \to H^q_I(R)\big).$$

Then $O_\bullet H^q_I(R)$ is a filtration on $H^q_I(R)$ compatible with the filtration on D_R and it is interesting to study how close $F_\bullet$ and $O_\bullet$ are.

Comparison with other filtrations

Since $\left(H^q_I(R), F_\bullet\right)$ underlies a mixed Hodge module supported on Z:

$$I \cdot F_p H^q_I(R) \subseteq F_{p-1} H^q_I(R) \quad \text{for all} \quad p.$$

We have also seen that $I \cdot F_0 H^q_I(R) = 0$, hence

$$F_p H^q_I(R) \subseteq O_p H^q_I(R) := \{u \in H^q_I(R) \mid I^{p+1} u = 0\}$$

If $q = \mathrm{codim}(I)$, then

$$O_p H^q_I(R) = E_p H^q_I(R) := \mathrm{Im}\big(\mathrm{Ext}^q_R(R/I^{p+1}, R) \to H^q_I(R)\big).$$

Then $O_\bullet H^q_I(R)$ is a filtration on $H^q_I(R)$ compatible with the filtration on D_R and it is interesting to study how close $F_\bullet$ and $O_\bullet$ are.
Remark. If $q > \mathrm{codim}(I)$ and $H^q_I(R) \neq 0$, then the $O_p H^q_I(R)$ are not finitely generated (Lyubeznik).

The case of locally complete intersections

Example 1 (Perlman-Raicu). If $Z \subset \mathbf{C}^{m\times n}$ is the subvariety of matrices of rank $\leq p$, with $m < n$, and $q = \mathrm{codim}(I)$, then $F_\bullet H^q_I(R) = O_\bullet H^q_I(R)$.

The case of locally complete intersections

Example 1 (Perlman-Raicu). If $Z \subset \mathbf{C}^{m\times n}$ is the subvariety of matrices of rank $\leq p$, with $m < n$, and $q = \mathrm{codim}(I)$, then $F_\bullet H^q_I(R) = O_\bullet H^q_I(R)$.
Example 2 (M.-Popa). If Z is a locally complete intersection, of pure codimension q, then $F_\bullet H^q_I(R) = O_\bullet H^q_I(R)$ if and only if Z is smooth.

The case of locally complete intersections

Example 1 (Perlman-Raicu). If $Z \subset \mathbf{C}^{m\times n}$ is the subvariety of matrices of rank $\leq p$, with $m < n$, and $q = \mathrm{codim}(I)$, then $F_\bullet H^q_I(R) = O_\bullet H^q_I(R)$.
Example 2 (M.-Popa). If Z is a locally complete intersection, of pure codimension q, then $F_\bullet H^q_I(R) = O_\bullet H^q_I(R)$ if and only if Z is smooth.

From now on, let Z be reduced and locally complete intersection of pure codimension q. Expect: in this case the behavior of the Hodge filtration on local cohomology should be related to other ways of measuring the singularities of Z.
Definition. The *singularity level* of the Hodge filtration on $H^q_I(R)$ is

$$p(Z) := \sup\{k \mid F_k H^q_I(R) = O_k H^q_I(R)\}$$

Remark. If Z is a hypersurface, this is the largest k such that the pair (X, Z) is $(k+1)$-log canonical.

The case of locally complete intersections, cont'd

We study $p(Z)$ by describing it in terms of the Hodge filtration on $R_{f_1 \cdots f_q}$, where $f_1, \ldots, f_q$ is a (local) general system of generators of I.

The case of locally complete intersections, cont'd

We study $p(Z)$ by describing it in terms of the Hodge filtration on $R_{f_1\cdots f_q}$, where $f_1,\ldots,f_q$ is a (local) general system of generators of I.

More precisely, we show that $p(Z)\geq k$ if and only if

$$\big(f_1^{b_1}\cdots f_q^{b_q}\mid 0\leq b_i\leq k, b_1+\ldots+b_q=k(q-1)\big)$$

$$\subseteq I_k(f_1\cdots f_q)+(f_1^{k+1},\ldots,f_q^{k+1})$$

in a neighborhood of Z.

The case of locally complete intersections, cont'd

Using properties that we know for the Hodge ideals of a hypersurface, we show that $p(Z)$ satisfies semicontinuity and doesn't increase under hyperplane sections. Moreover, if Z is singular, then

$$p(Z) \leq \frac{\dim(Z) - 1}{2}$$

The case of locally complete intersections, cont'd

Using properties that we know for the Hodge ideals of a hypersurface, we show that $p(Z)$ satisfies semicontinuity and doesn't increase under hyperplane sections. Moreover, if Z is singular, then

$$p(Z) \leq \frac{\dim(Z) - 1}{2}$$

Theorem 2 (M.-Popa). For $p \geq 0$, we have $p(Z) \geq p$ if and only if the canonical morphism $\Omega_Z^k \to \underline{\Omega}_Z^k$ is an isomorphism for $0 \leq k \leq p$.

Here: $\underline{\Omega}_Z^k$ is the (shifted) k-graded piece of the Du Bois complex of Z. The case $q = 1$ of above theorem: M-Olano-Popa-Witaszek and Jung-Kim-Saito-Yoon.