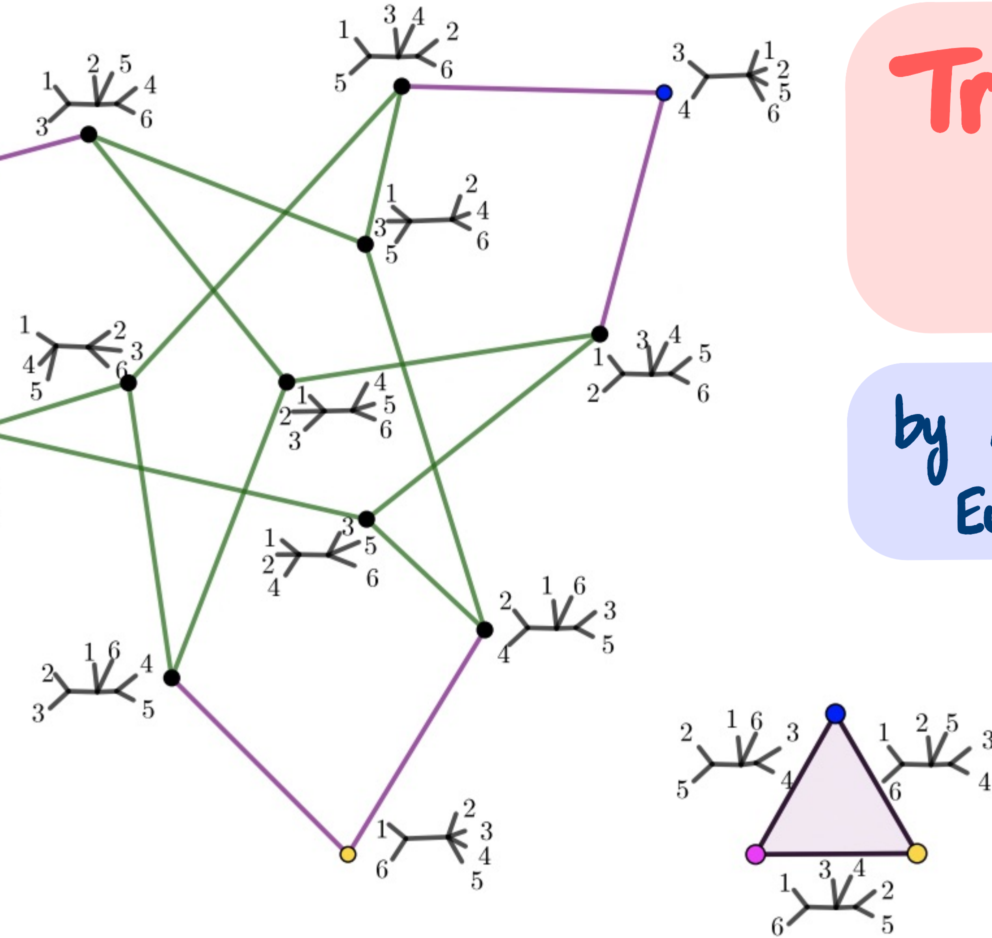
Tropical Flag Varieties

by Madeline Brandt, Chris Eur, and Leon Zhang



Example. $Gr(2,4)$ & $\mathcal{U}_{2,4}$

The Grassmannian $Gr(2,4) \hookrightarrow \mathbb{P}^{\binom{4}{2}-1}$ via the Plücker embedding:

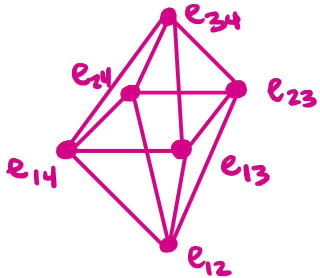
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \longmapsto \left(p_{ij} = \det \begin{bmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{bmatrix} : i,j \in \binom{[4]}{2} \right)$$

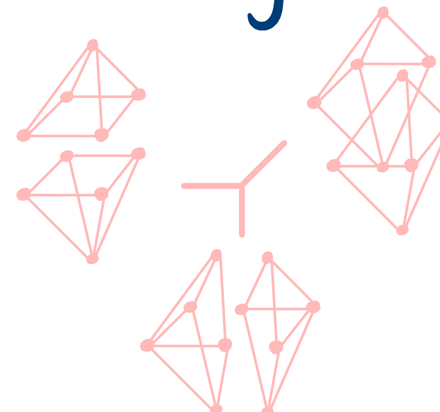
and is cut out by the equation $P_{14}P_{23} - P_{13}P_{24} + P_{12}P_{34}$.

Points in $Gr(2,4)^0$ are realizations of the uniform matroid.

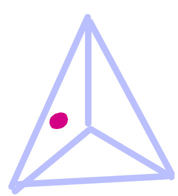
$\mathrm{trop}(Gr(2,4)^\circ)$ is + lineality space.

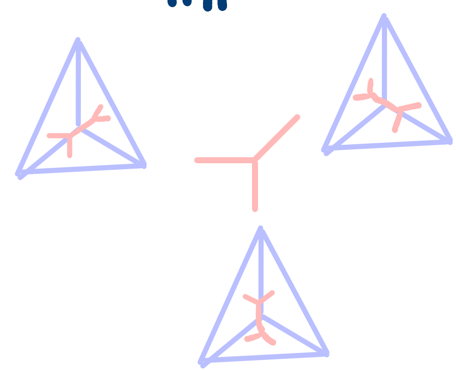
- $\mathrm{trop}(Gr(2,4)^\circ)$ is weight vectors inducing a matroidal subdivision of $B(M)$

$B(M) =$ 



- $\mathrm{trop}(Gr(2,4)^\circ)$ is tropical lines in $\mathbb{TP}^3$:





Dressians

M rank r matroid on $[n]$

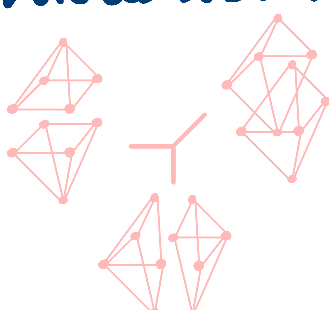
$Gr(r,n) \hookrightarrow \mathbb{P}^{\binom{n}{r}-1}$ cut out by Grassman-Plücker relations.

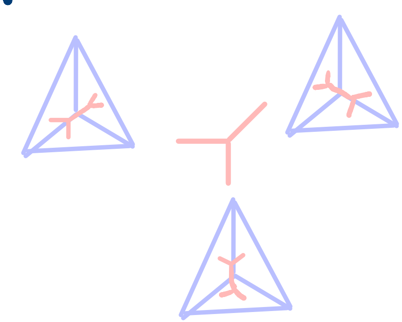
Setting variables indexing non-bases of M to 0 gives equations cutting out the points of $Gr(r,n)$ realizing M.

Theorem [Dress-Wentzel, Speyer, Hermann-Jensen-Joswig-Sturmfels]

The **Dressian of M** is:

- tropical prevariety
- matroidal subdivisions
- valuated matroids
- tropical linear spaces





Example. $(\mathcal{U}_{1,4}, \mathcal{U}_{2,4})$ ← flag matroid

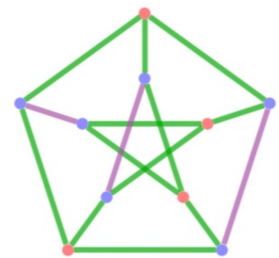
(A) $Fl(1,2;4) \hookrightarrow \mathbb{P}^3 \times \mathbb{P}^5$ cut out by:

$$\langle P_{14}P_{23} - P_{13}P_{24} + P_{12}P_{34},\quad P_4P_{23} - P_3P_{24} + P_2P_{34},\quad P_4P_{13} - P_3P_{14} + P_1P_{34},\quad P_4P_{12} - P_2P_{14} + P_1P_{24},\quad P_3P_{12} - P_2P_{13} + P_1P_{23}\rangle.$$

GP on $\mathcal{U}_{2,4}$

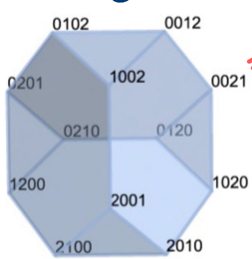
IP putting point $\subset$ line

tropical prevariety:



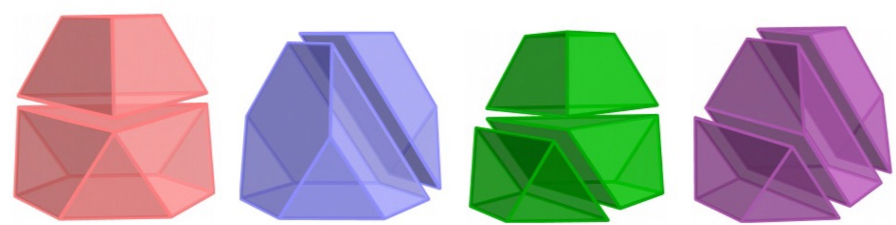
← flag Dressian

(C) Base Polytope:

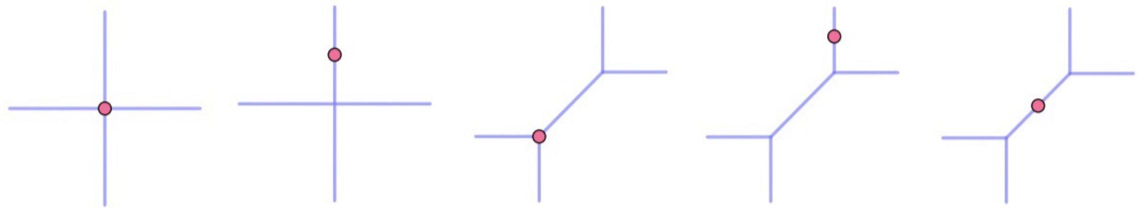


$= e_3 + e_{34}$

Subdivisions into flag matroid polytopes:



(D) tropical flags of a point in a line in $\mathbb{TP}^3$:



Flag Matroids

Definition. A *flag matroid* is a sequence of matroids

$$(M_1, \ldots, M_k)$$

of ranks $(r_1, \ldots, r_k)$ on $[n]$ such that every circuit of M_j is a union of circuits of M_i for $i < j$.

Example. $(U_{1,4}, U_{2,4})$ is a flag matroid of rank $(1,2)$ on $[4]$.

$\mathcal{C} = \binom{[4]}{2}$ $\mathcal{C} = \binom{[4]}{3}$

Valuated Flag Matroids

Definition. A *valuated flag matroid* is a flag matroid $(M_1, \dots, M_k)$ together with functions $(v_1, \dots, v_k)$,

$$v_i : B(M_i) \to \mathbb{R}$$

such that for $j \leq i$, $J \in B(M_j)$, $I \in B(M_i)$, and $e \in J \setminus I$, there exists $f \in I \setminus J$ such that

$$v(J) + v(I) \geq v(J \setminus \{e\} \cup \{f\}) + v(I \setminus \{f\} \cup \{e\}).$$

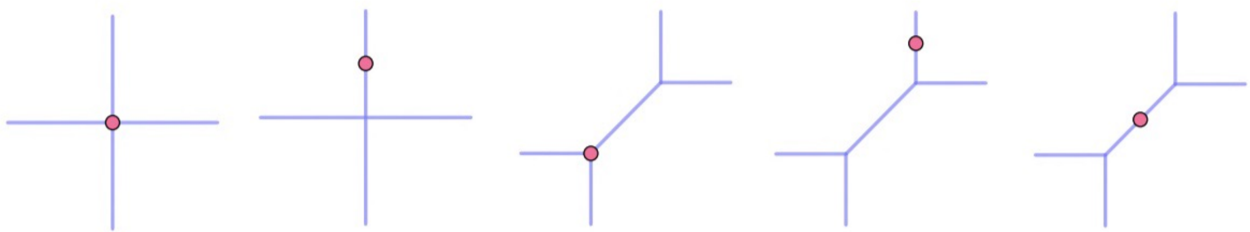
Tropical Linear Spaces (I)

Let (v, M) be a valuated matroid. (valuated flag matroid length 1)

This data gives a tropical linear space

$$\mathrm{trop}(v) = \bigcap_{\substack{T \text{ rank } r \\ |T| = r+1}} \left\{ u \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \text{the min in } \{u_j + v(T \setminus j)\}_{j \in T} \\ \text{is attained twice} \end{array} \right\}$$

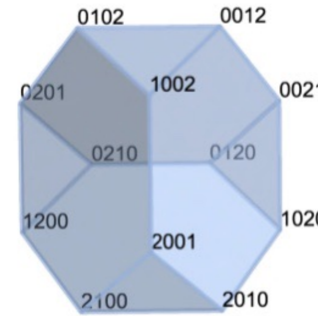
$\mathrm{trop}(v_1) \; \ldots \; \mathrm{trop}(v_k)$



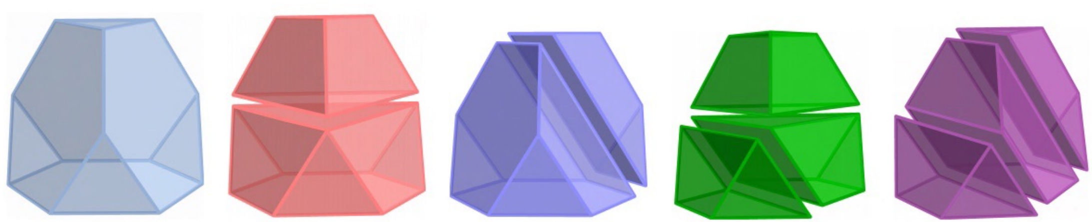
Base Polytopes of Flag Matroids (C)

$(M_1, \dots, M_k)$

$$\underbrace{\sum_{1 \le i \le k}}_{\text{Minkowski}} \underbrace{\operatorname{Conv}(e_B \mid B \in \mathcal{B}(M_i))}_{\text{Base polytope of } M_i} \subset \mathbb{R}^n$$



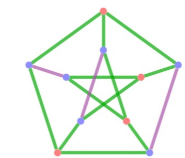
Weights: $(v_1, \dots, v_k) \in \mathbb{R}^{|\mathcal{B}(M_1)|} \times \dots \times \mathbb{R}^{|\mathcal{B}(M_k)|} \rightsquigarrow \Big(\underset{\text{vertex}}{P} \mapsto \min\{ v_1(B_1) + \dots + v_k(B_k) \mid P = e_{B_1} + \dots + e_{B_k} \} \Big)$



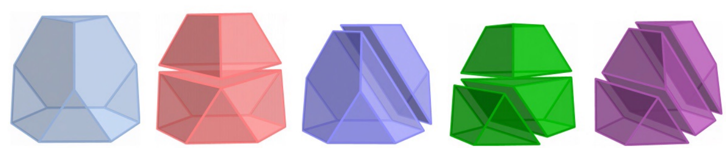
Main Theorem. The following are equivalent:

(A) $(v_1, \dots, v_k)$ is a point on a tropical prevariety called the Flag Dressian

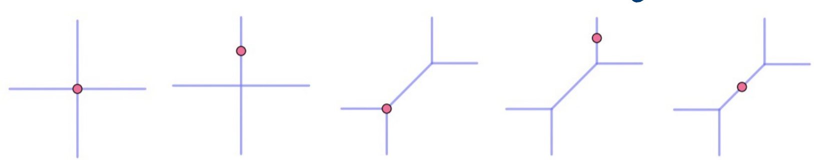
(B) $(v_1, \dots, v_k)$ is a valuated flag matroid on the underlying flag matroid $(M_1, \dots, M_k)$



(C) $(v_1, \dots, v_k)$ induces a subdivision of the base polytope of $(M_1, \dots, M_k)$ into base polytopes of flag matroids.



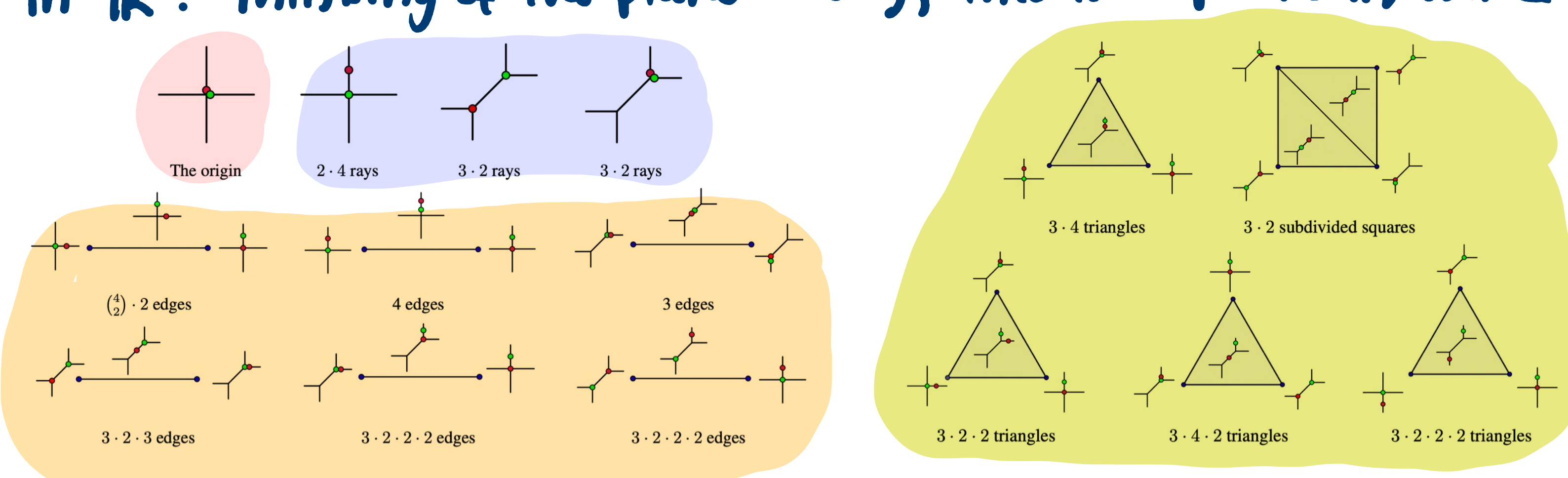
(D) the tropical linear spaces form a flag: $\mathrm{trop}(v_1) \subset \cdots \subset \mathrm{trop}(v_k)$



Realizibility

Theorem. Every valuated flag matroid on $n \leq 5$ is realizible. The tropicalization of $Fl(r_1, \ldots, r_k; n)$ equals the flag Dressian.

Example. $Fl(1,2,3;4)$. The tropicalization of $Fl(1,2,3;4)$ is the flag Dressian which parameterizes a point in a line in a plane in $\mathbb{R}^3$. Thinking of the plane dually, this is 2 points in a line.

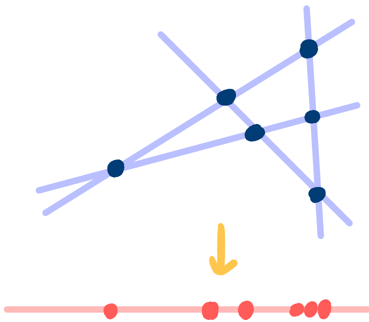


f-vector of $\text{trop}(Fl(1,2,3;4)) = (1, 20, 79, 78)$

Happy Friday
&
Thanks to the
organizers!!

Example [failure of realizibility for n=6].

Consider $(U_{2,6}, M_4)$:

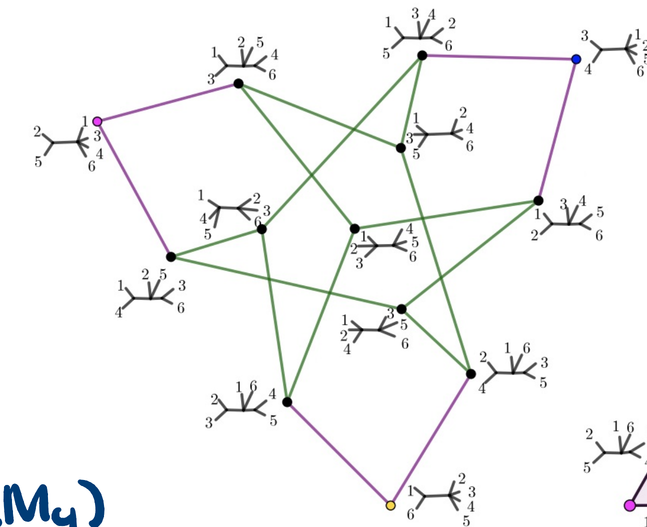


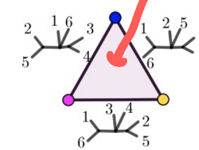
M_4

$\downarrow$

$U_{2,6}$

It's Dressian:

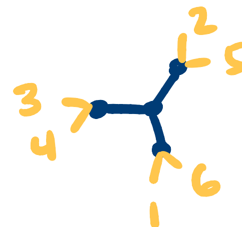


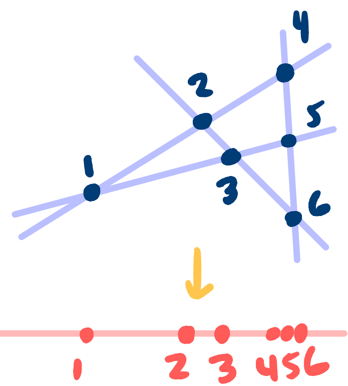


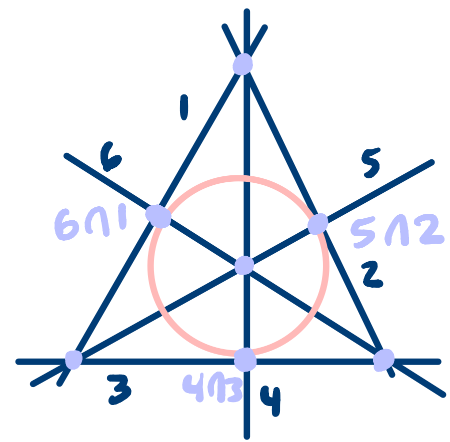
not in trop variety in char 0

All points in $Dr(U_{2,6})$ and $Dr(M_4)$ are realizable. So, points in the triangle correspond to two realizable valuated matroids that fail to form a realizable quotient (as pictured)

Points in the triangle have the tree:







So, over the residue field, we need

$1=6,\ 3=4,\ 2=5$

dually, this means finding a line intersecting at

$6\cap 1,\ 5\cap 2,\ 4\cap 3.$

(can only do in char $= 2$)

Theorem A:
proof

prevariety

valuations

$$\begin{tikzcd}
(A) \arrow[r, leftrightarrow, "\text{definitional}"] & (B) \arrow[dl, "\text{medium}"'] \arrow[d, leftrightarrow, "\text{not too bad}"] \\
(C) \arrow[r, "\text{hardest}"'] & (D)
\end{tikzcd}$$

subdivs

trop linear flags