Logarithmic Hilbert Scheme of Curves

April 16th 2021

ICERM CONFERENCE ON POLYTOPES & AG

w/ Davesh Maulik

"Logarithmic DT theory"

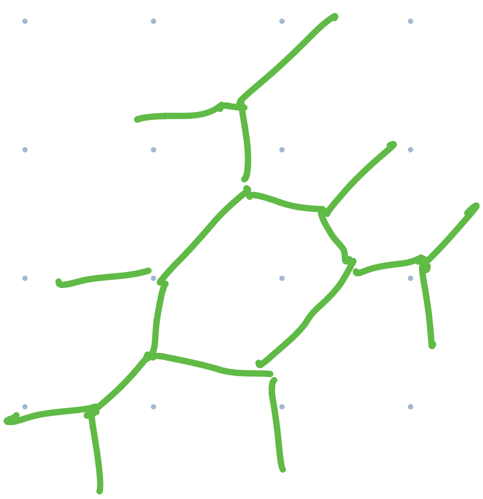
+ work in progress.

Two influential stories in enumerative geometry

12/'03

Tropical curve counting and correspondence theorems

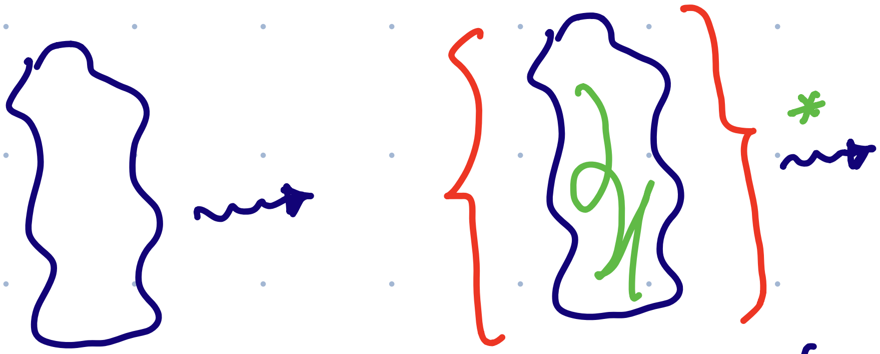
[Mikhalkin, Nishinou-Siebert, many others]



Gromov-Witten/Donaldson-Thomas correspondence

[Maulik-Nekrasov-Okounkov-Pandharipande and many others]

Enumerative Geometry: the game.



"CURVE COUNTING" INVARIANTS

X

CURVES IN X

$\mathcal{M}(X)$ w/ FIXED TOPOLOGY

* : "Integrate" natural cohomology classes against the "VIRTUAL" class.

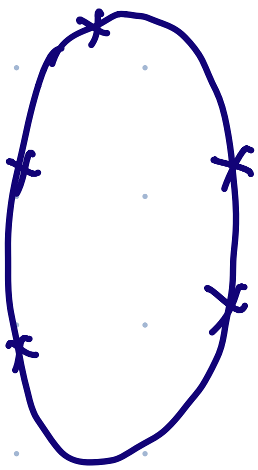
KEEP IN MIND: • $X \rightsquigarrow \mathcal{M}(X) \rightsquigarrow \chi_{top}(\mathcal{M}(X))$.

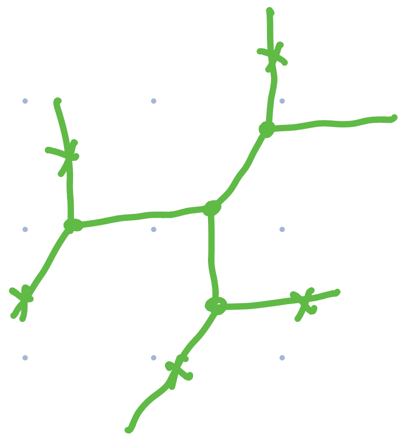
• Intersecting natural subvarieties

The <u>tropical</u> story (Mikhalkin & Mirror Symmetry)

N_d = # Degree d rational curves through $3d-1$ points in $\mathbb{P}^2$

N_d^{trop} = # Degree d rational tropical curves through $3d-1$ points in $\mathbb{R}^2$



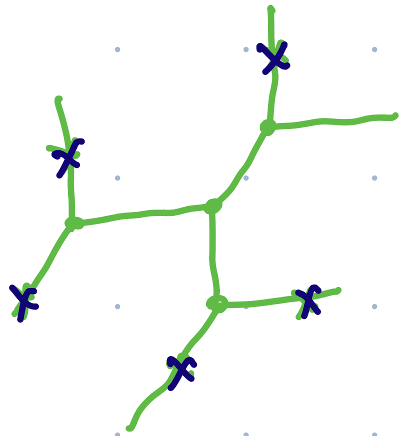


This has been proved, reproved in many nice ways, and generalized.

<u>VERSIONS:</u> $g=0$ X: toric

[limited positive genus applicability]

The basic geometry

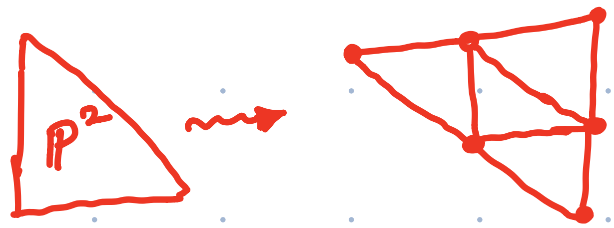


SECRETLY TWO PICTURES:

TORIC: 4 TORIC SURFACES GLUED ALONG BOUNDARY

TROPICAL: IMAGE OF A CURVE $C \subseteq \mathbb{G}_m^2$ under TROPICALIZATION

Mikhalkin: Degenerate



- Specialize the POINTS so they're "far away".

why?

- Decompose contributions over pieces

Around the same time... [MNOP]

The GW/DT correspondence.

Two ways of "counting curves" in X ← THREEFOLD.

ROUTE ONE: $\mathrm{Hilb}(X)_{\chi,\beta}$

⇝ The Donaldson–Thomas invariants $DT_{\chi,\beta}(X)$.

ROUTE TWO: $\overline{M}_g(X,\beta) = \{$  $\to X \}$

⇝ The Gromov–Witten invariants $GW_{g,\beta}(X)$

THE MNOP Conjectures:

$$\sum_g GW_{g,\beta}\, u^{2g-2} = \sum_{\chi} DT_{\chi,\beta}\, q^{\chi} \times \text{a Combinatorial Factor}$$

There is no way to match invariants one-by-one.

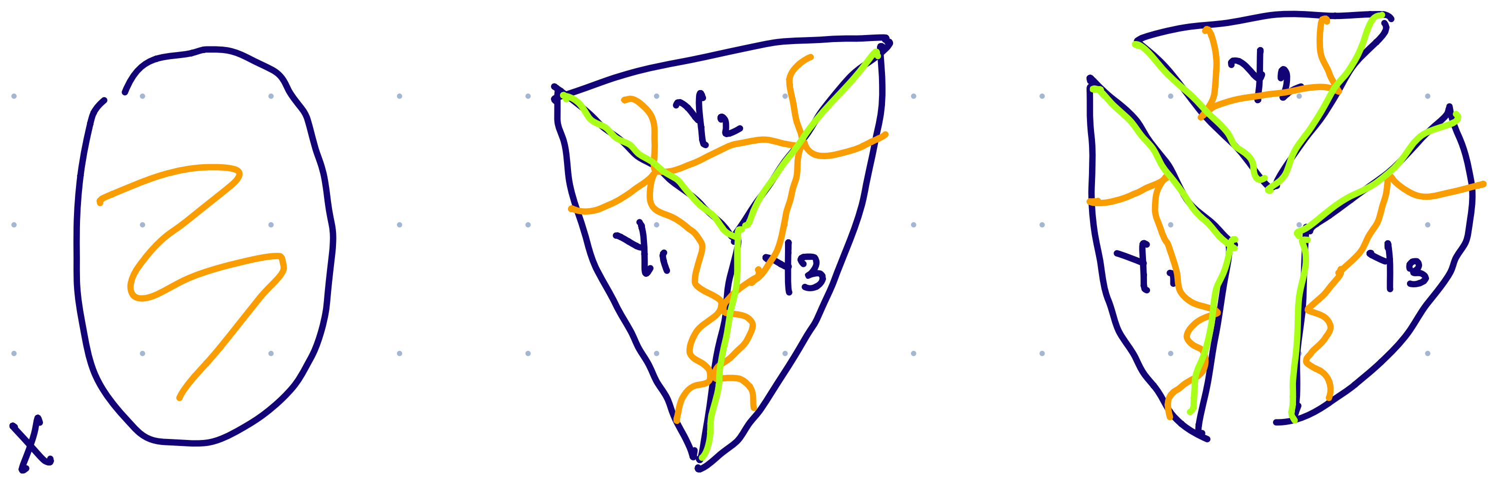
Remarks:

- Special to (surfaces and) threefolds
- No "geometric" reasoning. Proved in many cases with hard proofs [Pandharipande - Pixton; Maulik - Oblomkov - Okounkov - Pand.]
- PHYSICS

(ONE) MODERN VIEWPOINT ON TROPICAL COUNTING

[Abramovich–Chen
Gross–Siebert]

Degenerate the target X



• Form X^{trop} [dual complex]

• Decompose $GW_{g,\beta}(X) = \sum_{\gamma:\ \text{tropical curves}} GW_\gamma^{log}$ [ACGS]

• Glue $GW_\gamma = \ast_{\text{vertices } v \text{ of } \gamma} GW^{log}_{\gamma(v)}$ $\left\{ \begin{array}{l} \text{R '19} \\ \text{ACGS '20 Wu '21} \end{array} \right.$

GW ONLY!

A REMARK FOR MATROIDAL FRIENDS

THE DT SIDE:

A NEW HILBERT SCHEME:

X a toric variety

$$\mathrm{Hilb}(X)_{\chi,\beta} \supseteq \mathrm{Hilb}^{\circ}(X)_{\chi,\beta} \longrightarrow \text{Katz-Payne}$$

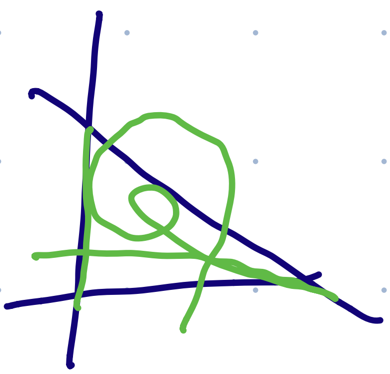
$$\| $$

$\{$ Subschemes Z in X that are tropical $\}$

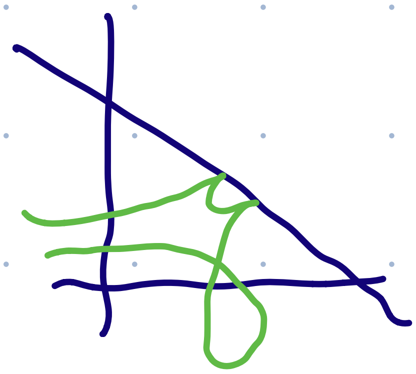
$\|$

- $Z \times T \longrightarrow X$ is flat

$\approx$ Z meets T-orbits in expected dimension.



NOT TROPICAL



TROPICAL.

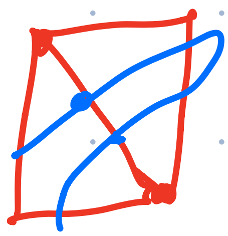
WHAT WE NEED

Not compact

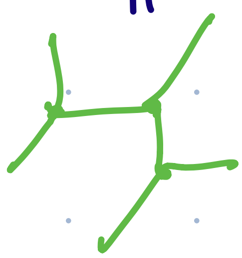
$$\mathrm{Hilb}(X)_{\chi,\beta} \neq \mathrm{LogHilb}(X)_{\chi,\beta} \supsetneq \mathrm{Hilb}^{\circ}(X)_{\chi,\beta}$$

Not tropical

compact by Tevelev, Ulirsch, Gubler + ϵ



Each point of $\mathrm{LogHilb}(X)_{\chi,\beta}$ gives us:

- A tropical curve  in X^{trop}

  Γ

- A subscheme $Z \hookrightarrow \mathcal{X}_\Gamma$ in the degeneration determined by Γ

Ambient Mathematics:

- T-orbit closures $G(k,n)$
- Gröbner degenerations.

THEOREM: there is a moduli space

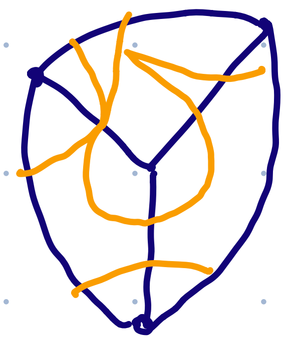
$$\begin{tikzcd} \mathcal{Z} \arrow[r] \arrow[dr] & \mathfrak{X} \arrow[r] \arrow[d] & X \\ & \mathrm{LogHilb}(X) & \end{tikzcd}$$

which is proper (& equipped with a virtual class)

such that at every point

$$\begin{tikzcd} \mathcal{Z}_p \arrow[r, hook] \arrow[dr] & \mathfrak{X}_p \arrow[d] \\ & P \end{tikzcd}$$

P in $\mathrm{LogHilb}(X)$

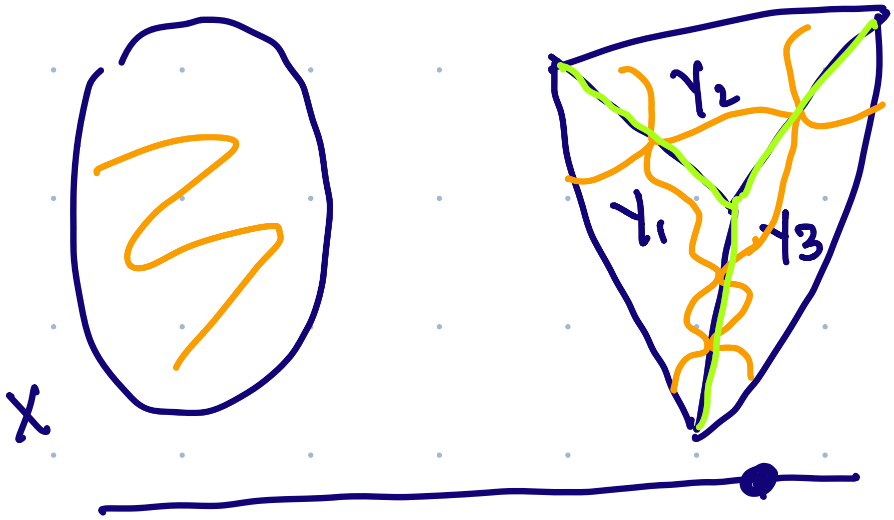


$\mathfrak{X}_p$.

is a tropical subscheme. LogHilb(X) breaks up into locally closed subsets of subschemes with a fixed tropicalization.

$\mathrm{Hilb}^\circ(X)_{\chi,\beta}$ is the principal component.

TROPICAL GEOMETRY & MNOP : the dream



GOAL: $GW^X_\beta(u) \longleftrightarrow DT^X_\beta(q)$

SUM OVER TROP's: $\sum_\gamma GW_\gamma(u) \longleftrightarrow \sum_\gamma DT_\gamma(q)$

FOR EACH γ: $GW_\gamma(u) \longleftrightarrow DT_\gamma(q)$

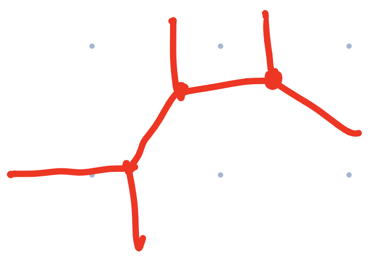
INSIDE THE VERTICES: $GW_v(u) \longleftrightarrow DT_v(q)$

vertices v of γ | vertices v of γ

Not reality yet...

PRECURSORS & EVIDENCE

In tropical curve counting



Γ $\rightsquigarrow$ multiplicity

$$n_\Gamma = \prod n_v \quad ; \quad n_v \in \mathbb{N}$$

Block–Göttsche

Replace $n_v \rightsquigarrow [n_v]_q$ || "quantum" $[n_v]_q$ a polynomial in q

Bousseau '19 / Parker '18 + $(GW/DT)^{\log}$:

The quantum counts are exactly our

$$DT_\gamma(q)$$

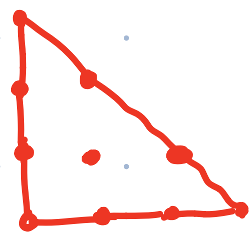
LOTS OF COOL WORK INTERPRETING THIS STUFF:

Mikhalkin, Nicaise–Payne–Schroeter, Filippini–Stoppa, Mandel, ...

A SPECIAL CASE: w/ the main idea.

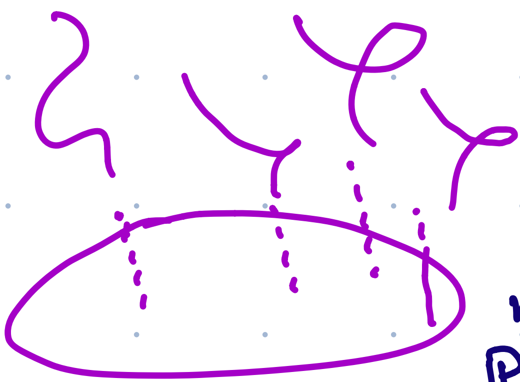
- Given a 2D lattice polytope $\mathcal{P}$ we get

$X_{\mathcal{P}}$. toric SURFACE



"Primary"

- The system $\mathbb{P}^{n_{\mathcal{P}}}$ of curves on $X_{\mathcal{P}}$



$\mathbb{P}^{n_{\mathcal{P}}}$: linear system.

- The secondary polytope $\mathcal{S}$ is the toric variety

$$\mathbb{P}_{\log} \longrightarrow \mathbb{P}^{n_{\mathcal{S}}} /\!\!/_{\text{Chow}} (\mathbb{C}^*)^2$$

[+ stack structure]

ORBIT

Gelfand – Kapranov – Zelivinsky: told us in the 1980's

[+ Sturmfels, Billera]

$$\mathbb{P}^{n_\delta} \neq \mathbb{P}_{\log} \cong \left\{ \begin{array}{c} \text{Curves in the linear} \\ \text{system that} \\ \text{are tropical} \end{array} \right\}$$

the linear system

But DT theory suggests questions: Among them:

SecPoly highly singular

But PP^*(SecPoly) plays a crucial role in the new log DT theory!

$$MW(\text{SecPoly}) \approx MW(\text{Hypersimplex})$$

$$MW(\text{Permutohedron}).$$

THANKS!