Families of Gröbner degenerations

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Motivation

X a projective variety, a *toric degeneration* of X is a flat morphism $\pi : \mathfrak{X} \to \mathbb{A}^d$ with generic fibre isomorphic to X and special fibre $\pi^{-1}(0)$ a toric variety.

Example: $\mathfrak{X} = V(xy - x^2 + ty^2) \subset \mathbb{P}^1_{x,y} \times \mathbb{A}^1_t$

Question: How are different toric degenerations of X related?

Have seen similar questions in other talks:

- Laura Escobar: How are different Newton–Okounkov polytopes related?
- Alfredo Nájera Chávez: Many toric degenerations of Grassmannians from cluster algebras.

Toric degenerations from valuations

$A = \bigoplus_{i \geq 0} A_i$ graded algebra and domain, $\mathfrak{v} : A \setminus \{0\} \to \mathbb{Z}^d$ a valuation with *image*$(\mathfrak{v})$ a finitely generated semigroup of rank $d := \dim_{\mathrm{Krull}} A$.

[Anderson] Exists a toric degeneration of *Proj*(A) with special fibre $TV(\Delta(A, \mathfrak{v}))$, where

$$\Delta(A, \mathfrak{v}) := \overline{\mathrm{conv}\left(\bigcup_{i \geq 1} \left\{ \frac{\mathfrak{v}(f)}{i} : f \in A_i \right\}\right)}$$

Newton–Okounkov polytope

A set $\{b_1, \ldots, b_n\} \subset A$ of algebra generators is a *Khovanskii basis* for $\mathfrak{v}$ if $\mathfrak{v}(b_1), \ldots, \mathfrak{v}(b_n)$ generate *image*$(\mathfrak{v})$.

Gröbner toric degenerations

Reminder: $f = x^2 + y \in \mathbb{C}[x, y]$ and $w = (1, 1)$, then $in_w(f) = y$ and for $J \subset \mathbb{C}[x_1, \ldots, x_n]$ ideal $in_w(J) := (in_w(f) : f \in J)$

$$Trop(J) := \{w \in \mathbb{R}^n : in_w(J) \not\ni \text{monomials}\}$$

Let $A := \mathbb{C}[x_1, \ldots, x_n]/J$ with J homogeneous prime ideal and $w \in Trop(J)$ such that $in_w(J)$ is binomial and prime (i.e. *toric*).

Then exists a flat family with generic fibre $Proj(A)$ and special fibre the toric variety $Proj(\mathbb{C}[x_1, \ldots, x_n]/in_w(J))$, called a *Gröbner toric degeneration*.

Motivating result

Theorem (Kaveh–Manon, B.)

Let A be a positively graded algebra and domain, $\mathfrak{v} : A \setminus \{0\} \to \mathbb{Z}^d$ full rank valuation with finitely generated value semigroup. Then there exists an isomorphism of graded algebras

$$\mathbb{C}[x_1, \dots, x_n]/J \cong A$$

such that the toric variety of the Newton–Okounkov polytope is isomorphic to the toric variety of a Gröbner toric degeneration for some $w \in Trop(J)$:

$$TV(\Delta(A, \mathfrak{v})) \cong Proj(\mathbb{C}[x_1, \dots, x_n]/in_w(J)).$$

Motivating result

Idea of Proof: Choose a finite Khovanskii basis $b_1, \ldots, b_n \in A$. Take

$$\pi : \mathbb{C}[x_1, \ldots, x_n] \to A, \quad x_i \mapsto b_i$$

and $J := \ker(\pi)$. Then by [B, Main Theorem] exits $w_\mathfrak{v} \in \text{Trop}(J)$ such that

$$in_{w_\mathfrak{v}}(J) \quad \text{is toric} \quad \Leftrightarrow \quad image(\mathfrak{v}) \quad \text{is finitely generated.}$$

Moreover, $\mathbb{C}[image(\mathfrak{v})] \cong \mathbb{C}[x_1, \ldots, x_n]/in_{w_\mathfrak{v}}(J)$. ■

Question: Given a family of full rank valuations with finitely generated value semigroups, can we find *one* ideal J that works for all valuations?

Recall: $\mathfrak{v}$ defines a filtration on A: $F_{m;\mathfrak{v}} := \{f \in A : \mathfrak{v}(f) \leq m\}$ for all $m \in \mathbb{Z}^d$ and $\leq$ a fixed total order. A vector space basis $\mathbb{B}$ of A is *adapted* to $\mathfrak{v}$ if $\mathbb{B} \cap F_{m;\mathfrak{v}}$ is a vector space basis for each $F_{m;\mathfrak{v}}$.

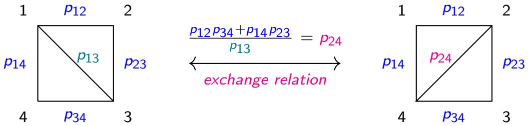
Example: Grassmannian cluster algebra

Let $A_{k,n}$ be the homogeneous coordinate ring of $Gr(k, \mathbb{C}^n)$ under its Plücker embedding.

[Scott] $A_{k,n}$ is a cluster algebra, i.e. recursively generated by

1. *seeds*: maximal sets of algebraically independent algebra generators, its elements are called *cluster variables (c.v.'s)*;
2. *mutation*: an operation to create a new seed from a given one by replacing one element.

Example: $A_{2,4} = \mathbb{C}[p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}]/(p_{13}p_{24} = p_{12}p_{34} + p_{14}p_{23})$



$s = \{p_{12}, p_{23}, p_{34}, p_{14}, p_{13}\}$ $\quad$ $s' = \{p_{12}, p_{23}, p_{34}, p_{14}, p_{24}\}$

$s_{\text{mut}} = \{p_{13}\}$ $\quad$ $s'_{\text{mut}} = \{p_{24}\}$

Example: Grassmannian cluster algebra

Alfredo's talk: $\forall$ seed s of $A_{k,n}$ $\exists$ full rank valuation $g_s : A_{k,n} \setminus \{0\} \to \mathbb{Z}^d$ and the ϑ-basis is adapted to all of them simultaneously. Let $A_{k,n}^{\text{prin},s}$ be the flat $\mathbb{C}[t_x : x \in s_{\text{mut}}]$-algebra defining the toric degeneration.

[Fomin–Zelevinsky]/[Reading]: $\exists$ flat $\mathbb{C}[t_x : x \text{ m.c.v.}]$-algebra $A_{k,n}^{\text{univ}}$ and projections $pr_s : \mathbb{C}[t_x : x \text{ m.c.v}] \to \mathbb{C}[t_x : x \in s_{\text{mut}}]$ for all seeds s that extend to

$$pr_s : A_{k,n}^{\text{univ}} \to A_{k,n}^{\text{prin},s}$$

called *coefficient specialization*.

Example: $A_{2,4}^{\text{prin},s} = \mathbb{C}[t_{13}][p_{12}, p_{23}, p_{34}, p_{14}, p_{13}, p_{24}]/(p_{13}p_{24} = t_{13}p_{12}p_{34} + p_{14}p_{23})$,
$A_{2,4}^{\text{univ}} = \mathbb{C}[t_{13}, t_{24}][p_{12}, p_{23}, p_{34}, p_{14}, p_{13}, p_{24}]/(p_{13}p_{24} = t_{13}p_{12}p_{34} + t_{24}p_{14}p_{23})$.

For each seed s we can apply the Motivating Theorem and get an ideal J_s and a Gröbner toric degeneration of J_s corresponding to $A_{k,n}^{\text{prin},s}$.

Question: How are different J_s related and what is $A_{k,n}^{\text{univ}}$ in this context?

Gröbner fan and standard monomial bases

Definition

For a homogeneous ideal $J \subset \mathbb{C}[x_1, \dots, x_n]$ its *Gröbner fan* $GF(J)$ is $\mathbb{R}^n$ with fan structure defined by

$$v, w \in C^\circ \quad \Leftrightarrow \quad \mathrm{in}_v(J) = \mathrm{in}_w(J).$$

Notation: $\mathrm{in}_C(J) := \mathrm{in}_w(J), w \in C^\circ$ and $A_C := \mathbb{C}[x_1, \dots, x_n]/in_w(J)$

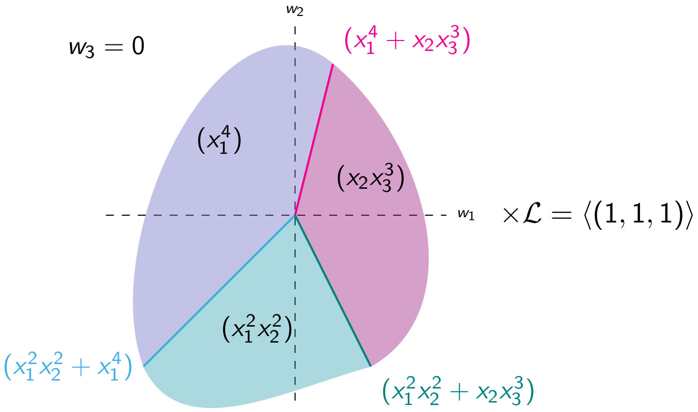
For $C \in GF(J)$ a maximal cone $\mathrm{in}_C(J)$ is generated by monomials. For every face $\tau \subseteq C$ we define

$$\mathbb{B}_{C,\tau} := \{\bar{\mathbf{x}}^\alpha \in A_\tau \mid \mathbf{x}^\alpha \notin \mathrm{in}_C(J)\}.$$

Then $\mathbb{B}_{C,\tau}$ is a vector space basis for A_τ called *standard monomial basis*.

Example

Take $J = (x_1^2x_2^2 + x_1^4 + x_2x_3^3) \subset \mathbb{C}[x_1, x_2, x_3]$. Then $GF(J)$ is $\mathbb{R}^3$ with the fan structure:



E.g. $\mathbb{B}_{\langle r_1, r_2\rangle} = \{\bar{\mathbf{x}}^a : x_2x_3^3 \not| \mathbf{x}^a\}$ gives a basis for A, A_{r_1}, A_{r_2} and $A_{\langle r_1, r_2\rangle}$.

Family of ideals

Let $C \in GF(J)$ be a maximal cone and choose $r_1, \dots, r_m$ representatives of primitive ray generators of $\overline{C} \in \mathrm{GF}(J)/\mathcal{L}$. Let $\mathbf{r}$ be the matrix with rows $r_1, \dots, r_m$. Define for $f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha \mathbf{x}^\alpha \in J$

$$\mu(f) := (\min_{c_\alpha \neq 0} \{r_1 \cdot \alpha\}, \dots, \min_{c_\alpha \neq 0} \{r_m \cdot \alpha\}) \in \mathbb{Z}^m.$$

In $\mathbb{C}[t_1, \dots, t_m][x_1, \dots, x_n]$ the *lift* of f is

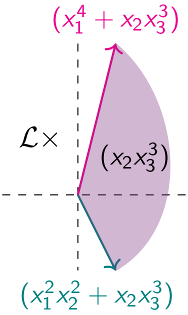
$$\tilde{f} := f(\mathbf{t}^{\mathbf{r}\cdot e_1} x_1, \dots, \mathbf{t}^{\mathbf{r}\cdot e_n} x_n)\mathbf{t}^{-\mu(f)} = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha \mathbf{x}^\alpha \mathbf{t}^{\mathbf{r}\cdot\alpha - \mu(f)}.$$

Definition/Proposition

The *lifted ideal* $\tilde{J} := (\tilde{f} : f \in J) \subset \mathbb{C}[t_1, \dots, t_m][x_1, \dots, x_n]$ is generated by $\{\tilde{g} : g \in \mathcal{G}\}$, where $\mathcal{G}$ is a *Gröbner basis* for J and C.

Example

Take $f = x_1^2x_2^2 + x_1^4 + x_2x_3^3 \in \mathbb{C}[x_1, x_2, x_3]$ and consider in $GF((f))$ the maximal cone C spanned by the rows of $r := \begin{pmatrix} 1 & 4 & 0 \\ 1 & -2 & 0 \end{pmatrix}$ and $\mathcal{L}$. We compute



$$\begin{aligned} \tilde{f}(t_1, t_2) &= f(t_1t_2x_1, t_1^4t_2^{-2}x_2, x_3)t_1^{-4}t_2^2 \\ &= t_1^6x_1^2x_2^2 + t_2^6x_1^4 + x_2x_3^3 \end{aligned}$$

- $\tilde{f}(0,0) = x_2x_3^3 = \text{in}_C(f)$,
- $\tilde{f}(0,1) = x_1^4 + x_2x_3^3 = \text{in}_{r_1}(f)$,
- $\tilde{f}(1,0) = x_1^2x_2^2 + x_2x_3^3 = \text{in}_{r_2}(f)$,
- $\tilde{f}(1,1) = f$.

Theorem

Let $\tilde{A} := \mathbb{C}[t_1, \dots, t_m][x_1, \dots, x_n]/\tilde{J}$ and recall $A_\tau = \mathbb{C}[x_1, \dots, x_n]/\mathrm{in}_\tau(J)$.

Theorem (B.–Mohammadi–Nájera Chávez)

$\tilde{A}$ is a free $\mathbb{C}[t_1, \dots, t_m]$-module with basis $\mathbb{B}_C$ and so the morphism

$$\pi : Spec(\tilde{A}) \to \mathbb{A}^m$$

is flat. In particular, π defines a flat family with generic fiber $Spec(A)$ and for every face $\tau \subseteq C$ there exists $\mathbf{a}_\tau \in \mathbb{A}^m$ and a special fiber $\pi^{-1}(\mathbf{a}_\tau) \cong Spec(A_\tau)$.

Example: $\tilde{A} = \mathbb{C}[t_1, t_2][x_1, x_2, x_3]/(t_1^6 x_1^2 x_2^2 + t_2^6 x_1^4 + x_2 x_3^3)$.

Toric degenerations

The tropicalization $Trop(J)$ is a subfan of $GF(J)$ of dimension $\dim_{\text{Krull}} A$.

Corollary (B.–Mohammadi–Nájera Chávez)

Consider the fan $\Sigma := C \cap Trop(J)$. If there exists $\tau \in \Sigma$ with $in_\tau(J)$ binomial and prime, then the family

$$\pi : Spec(\tilde{A}) \to \mathbb{A}^m$$

contains toric fibers isomorphic to $Spec(A_\tau)$ (affine toric scheme) and the standard monomials $\mathbb{B}_C$ are a basis for A_τ.

Question: How are $\tilde{A}$ and $A_{k,n}^{\text{univ}}$ related?

Back to cluster algebras

A cluster algebra A is of *finite type* if it has finitely many seeds. Such A has finitely many cluster variables $x_1, \ldots, x_N$ and a presentation

$$\mathbb{C}[x_1, \ldots, x_N]/J \cong A$$

where J is the saturation of the ideal generated by exchange relations. [Fomin–Williams–Zelevinsky, §6.8]

Grassmannians: $A_{k,n}$ is of finite type for $k = 2$ or $k = 3$ and $n \in \{6, 7, 8\}$. For $A_{2,n}$ all cluster variables are Plücker coordinates and $J = J_{2,n}$ is the Plücker ideal.

The cluster variables of $A_{3,6}$ are Plücker coordinates and two more, so

$$A_{3,6} \cong \mathbb{C}[X, Y, p_{123}, \ldots, p_{456}]/J_{3,6}$$

and $J_{3,6} \cap \mathbb{C}[p_{123}, \ldots, p_{456}]$ is the Plücker ideal.

Theorem for Grassmannians

Reminder: $J \subset \mathbb{R}[\mathbf{x}]$ is *totally non-negative* if $J \cap \mathbb{R}_{\geq 0}[\mathbf{x}] = \varnothing$ and

$$Trop^+(J) := \{w \in Trop(J) : in_w(J) \text{ totally non-negative}\}.$$

Theorem (B.–Mohammadi–Nájera Chávez)

For $(k,n) \in \{(2,n),(3,6)\}$ there exists a unique maximal cone C in the Gröbner fan of $J_{k,n}$ such that

1. *we have a canonical isomorphism $\tilde{A}_{k,n} \cong A^{\text{univ}}_{k,n}$ identifying universal coefficients with rays of C;*
2. *the standard monomial basis $\mathbb{B}_C$ for $A_{k,n}$ (and $\tilde{A}$) coincides with the ϑ-basis for $A_{k,n}$ (and $A^{\text{univ}}_{k,n}$);*
3. *$C \cap Trop(J_{k,n}) = Trop^+(J_{k,n})$ and there's a bijection between seeds s and maximal cones $\tau_s \in C \cap Trop(J_{k,n})$ such that the toric variety $TV(\Delta(A_{k,n}, g_s))$ is isomorphic to $Proj(A_{\tau_s})$.*

The Stanley–Reisner ideal of the cluster complex

For A a cluster algebra of finite type, seeds are maximal simplicies in a simplicial complex called *cluster complex* $\Delta(A)$. Its *Stanley–Reisner ideal* is

$$\mathcal{I}_{\Delta(A)} = (x_{i_1} \cdots x_{i_t} : \{x_{i_1}, \dots, x_{i_t}\} \not\subset s \ \forall s \text{ seed}) \subset \mathbb{C}[x_1, \dots, x_N]$$

Corollary (B.–Mohammadi–Nájera Chávez, Ilten)

For $(k,n) \in \{(2,n),(3,6)\}$, $A_{k,n}, J_{k,n}$ and C as in the Theorem. Then

1. *$in_C(J_{k,n}) = \mathcal{I}_{\Delta(A_{k,n})}$ and $C \cap Trop(J_{k,n})$ is a geometric realization of the cluster complex;*
2. *$\Delta(A_{k,n})$ is unobstructed [Ilten et al.], so $Proj(\mathbb{C}[\mathbf{x}]/\mathcal{I}_{\Delta(A_{k,n})})$, $Gr(k,\mathbb{C}^n)$ and $TV(\Delta(A_{k,n}, g_s))$ for all seeds s lie on the same component of their Hilbert scheme.*

Final remarks

1. For finite type cluster algebras of rank 2 many results still hold (all but the identification with A^{univ} and the unobstructedness).
2. [Escobar–Harada] For $\mathrm{Gr}(2,\mathbb{C}^n)$ the *flip* of Newton–Okounkov polytopes is the algebraic wall-crossing.

   From the cluster structure we obtain *tropical mutation maps* which coincide with the flip in this case.
3. [Kaveh–Manon] construct *toric families* $\psi : \mathfrak{X} \to TV(\Sigma)$ from subfans $\Sigma \subset Trop(J)$: for $\tau \in \Sigma$ and any $p_\tau \in \mathcal{O}_\tau \subset TV(\Sigma)$ we have $\psi^{-1}(p_\tau) \cong Spec(A_\tau)$.

   If $\Sigma \subset C$ for $C \in GF(J)$ maximal with m rays, then our flat family is the pull back of theirs along the universal torsor $\mathbb{A}^m \to X_\Sigma$ (Cox construction).

Thank you!

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A minimal generating set of $J_{3,6} \subset \mathbb{C}[p_{123}, \ldots, p_{456}, X, Y]$:

$$
\begin{aligned}
&p_{145}p_{236} - p_{123}p_{456} - X, && p_{124}p_{356} - p_{123}p_{456} - Y,\\
&p_{136}p_{245} - p_{126}p_{345} - X, && p_{125}p_{346} - p_{126}p_{345} - Y,\\
&p_{146}p_{235} - p_{156}p_{234} - X, && p_{134}p_{256} - p_{156}p_{234} - Y,\\
&p_{246}p_{356} - p_{346}p_{256} - p_{236}p_{456}, && p_{245}p_{356} - p_{345}p_{256} - p_{235}p_{456},\\
&p_{146}p_{356} - p_{346}p_{156} - p_{136}p_{456}, && p_{145}p_{356} - p_{345}p_{156} - p_{135}p_{456},\\
&p_{245}p_{346} - p_{345}p_{246} - p_{234}p_{456}, && p_{235}p_{346} - p_{345}p_{236} - p_{234}p_{356},\\
&p_{145}p_{346} - p_{345}p_{146} - p_{134}p_{456}, && p_{135}p_{346} - p_{345}p_{136} - p_{134}p_{356},\\
&p_{146}p_{256} - p_{246}p_{156} - p_{126}p_{456}, && p_{145}p_{256} - p_{245}p_{156} - p_{125}p_{456},\\
&p_{136}p_{256} - p_{236}p_{156} - p_{126}p_{356}, && p_{135}p_{256} - p_{235}p_{156} - p_{125}p_{356},\\
&p_{235}p_{246} - p_{245}p_{236} - p_{234}p_{256}, && p_{145}p_{246} - p_{245}p_{146} - p_{124}p_{456},\\
&p_{136}p_{246} - p_{236}p_{146} - p_{126}p_{346}, && p_{134}p_{246} - p_{234}p_{146} - p_{124}p_{346},\\
&p_{125}p_{246} - p_{245}p_{126} - p_{124}p_{256}, && p_{134}p_{245} - p_{234}p_{145} - p_{124}p_{345},\\
&p_{135}p_{245} - p_{235}p_{145} - p_{125}p_{345}, && p_{135}p_{236} - p_{235}p_{136} - p_{123}p_{356},\\
&p_{134}p_{236} - p_{234}p_{136} - p_{123}p_{346}, && p_{125}p_{236} - p_{235}p_{126} - p_{123}p_{256},\\
&p_{124}p_{236} - p_{234}p_{126} - p_{123}p_{246}, && p_{134}p_{235} - p_{234}p_{135} - p_{123}p_{345},\\
&p_{124}p_{235} - p_{234}p_{125} - p_{123}p_{245}, && p_{135}p_{146} - p_{145}p_{136} - p_{134}p_{156},\\
&p_{125}p_{146} - p_{145}p_{126} - p_{124}p_{156}, && p_{125}p_{136} - p_{135}p_{126} - p_{123}p_{156},\\
&p_{124}p_{136} - p_{134}p_{126} - p_{123}p_{146}, && p_{124}p_{135} - p_{134}p_{125} - p_{123}p_{145},
\end{aligned}
$$

$$f = p_{135}p_{246} - p_{156}p_{234} - Y - p_{123}p_{456} - X - p_{126}p_{345}.$$

The reduced Gröbner basis of $J_{3,6}$ for C consists contains the minimal generating set and additionally contains the following elements:

$$
\begin{aligned}
&p_{235}Y - p_{125}p_{234}p_{356} - p_{123}p_{256}p_{345}, && p_{134}X - p_{136}p_{145}p_{234} - p_{123}p_{146}p_{345},\\
&p_{146}Y - p_{124}p_{156}p_{346} - p_{126}p_{134}p_{456}, && p_{256}X - p_{156}p_{236}p_{245} - p_{126}p_{235}p_{456},\\
&p_{136}Y - p_{123}p_{156}p_{346} - p_{126}p_{134}p_{356}, && p_{346}X - p_{136}p_{234}p_{456} - p_{146}p_{236}p_{345},\\
&p_{245}Y - p_{125}p_{234}p_{456} - p_{124}p_{256}p_{345}, && p_{125}X - p_{123}p_{156}p_{245} - p_{126}p_{145}p_{235},\\
&p_{145}Y - p_{125}p_{134}p_{456} - p_{124}p_{156}p_{345}, && p_{124}X - p_{126}p_{145}p_{234} - p_{123}p_{146}p_{245},\\
&p_{236}Y - p_{126}p_{234}p_{356} - p_{123}p_{256}p_{346}, && p_{356}X - p_{136}p_{235}p_{456} - p_{156}p_{236}p_{345},\\
&p_{135}Y - p_{125}p_{134}p_{356} - p_{123}p_{156}p_{345}, && p_{135}X - p_{136}p_{145}p_{235} - p_{123}p_{156}p_{345},\\
&p_{246}Y - p_{124}p_{256}p_{346} - p_{126}p_{234}p_{456}, && p_{246}X - p_{146}p_{236}p_{245} - p_{126}p_{234}p_{456}.
\end{aligned}
$$

$$g = XY - p_{123}p_{156}p_{246}p_{345} - p_{126}p_{135}p_{234}p_{456} - p_{126}p_{156}p_{234}p_{345} - p_{123}p_{156}p_{234}p_{456} - p_{123}p_{126}p_{345}p_{456}.$$