On Newton–Okounkov bodies associated to Grassmannians

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joint with Lara Bossinger, Man-Wai Cheung and Timothy Magee

Outline

- Grassmannians $\mathrm{Gr}_k(\mathbb{C}^n)$ are varieties that admit the two possible types of cluster structures, namely $\mathcal{A}$ and $\mathcal{X}$.
- Rietsch-Williams (RW) used $\mathcal{X}$ cluster structure to construct Newton-Okounkov bodies and toric degenerations.
- Gross-Hacking-Keel-Kontsevich (GHKK) construct compactifications and toric degenerations from $\mathcal{A}$ cluster structure.

Goal: Explain how to get Newton-Okunkov bodies from cluster structures. In particular, explain how these approaches are related and draw some consequences.

The initial data

- A quiver Q is a finite directed graph without loops $\overset{\curvearrowright}{\bullet}$ nor 2-cycles $\bullet \rightleftarrows \bullet$
- $b_{ij} = \#\{\text{arrows } i \to j\} - \#\{\text{arrows } j \to i\}$
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- $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$
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- $s^\vee = (f_1, \dots, f_r)$ a $\mathbb{Z}$-basis of $M \rightsquigarrow \mathbb{C}(z^{f_1}, \dots, z^{f_r}) = \mathbb{C}(T_N)$

Cluster transformations

Let k be a vertex of Q and set $v_k = \sum_i b_{ik} f_i$. We have two cluster transformations $\mu_k^{\mathcal{A}}$ and $\mu_k^{\mathcal{X}}$.

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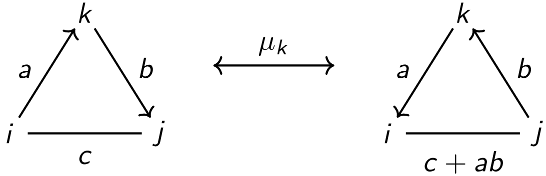
Key property: Mutations preserve the canonical volume form

$$\omega = \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_r}{z_r}$$

We think of these cluster transformations as gluing data.

Quiver mutation

We obtain a new quiver $\mu_k(Q)$ by performing a local transformation on Q around the vertex k



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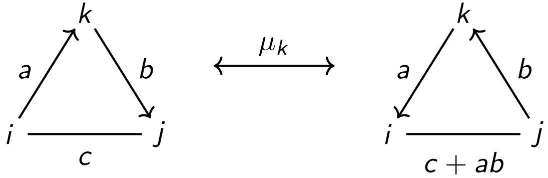


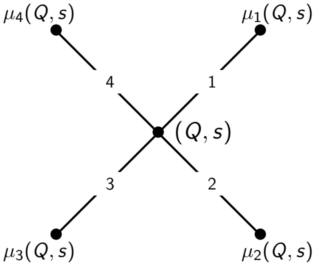
We also have the basis mutation $\mu_k(s) = (e'_1, \ldots e'_r)$, where

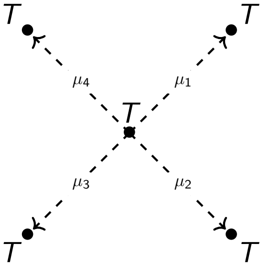
$$e'_i := \begin{cases} e_i + [\epsilon_{ik}]_+ e_k & i \neq k \\ -e_k & i = k \end{cases}$$

This formula also induces a mutation rule for the dual basis

$$\mu_k(s^*) = \mu_k(s)^*$$

Construction of cluster varieties via the r-regular tree $\mathbb{T}^r$





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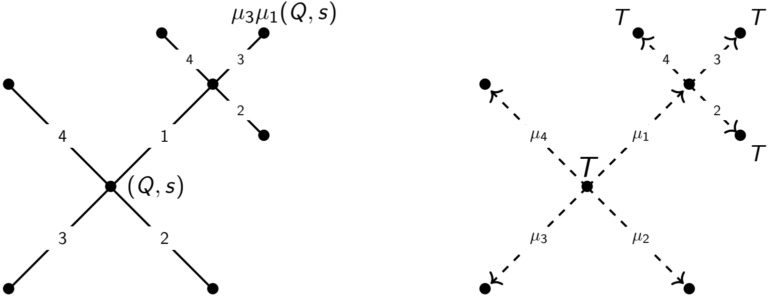

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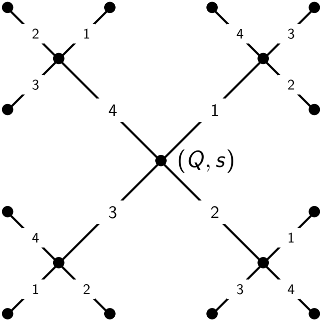

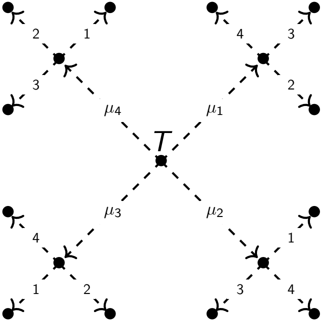

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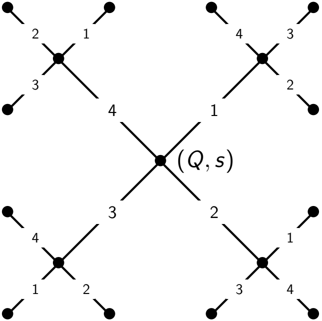

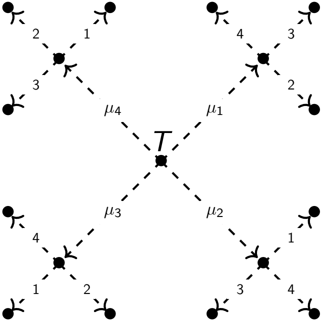

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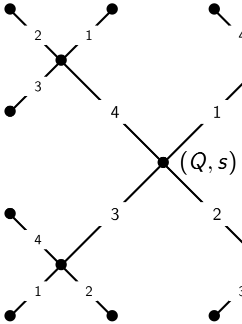

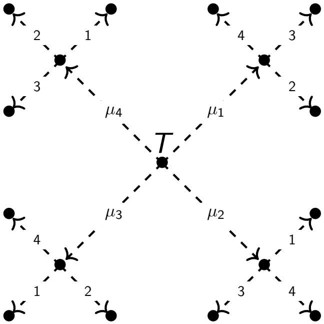

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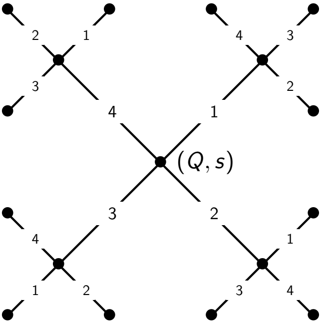

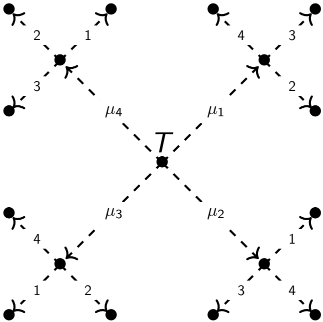

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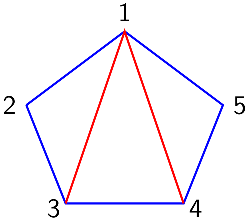
We allow to have frozen directions in which we do not mutate.

Example

A triangulation v of a pentagon defines:

- a torus in the affine cone $C(\mathrm{Gr}_2(\mathbb{C}^5))$



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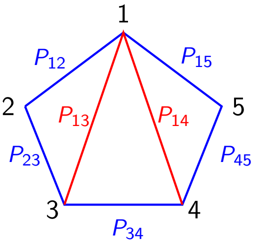

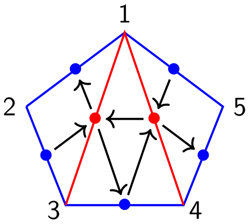

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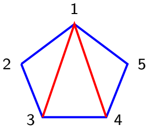
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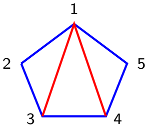
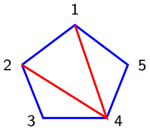
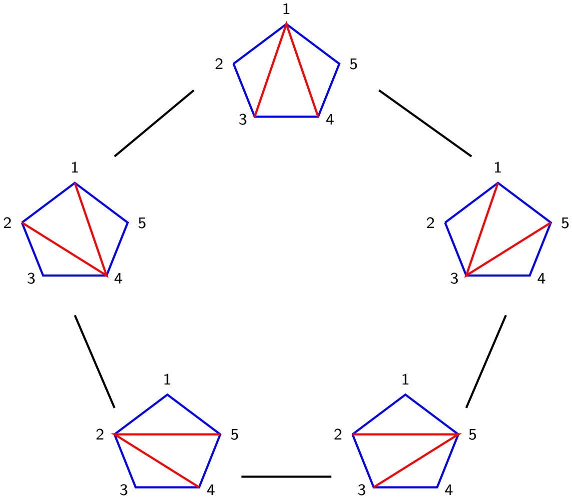
$$P_{24} = \frac{P_{23}P_{14} + P_{15}P_{12}}{P_{13}}$$



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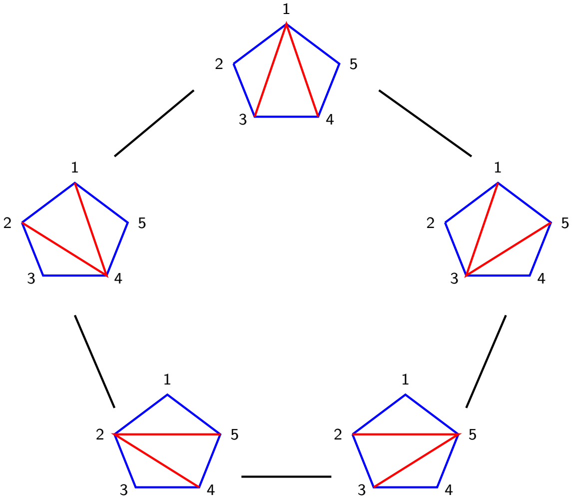

We obtain a cluster structure on the open positroid variety

$$X = \mathrm{Gr}_2(\mathbb{C}^5) \setminus V(P_{12}P_{23}P_{34}P_{45}P_{15})$$

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- $D_i = V(P_{[i,k+i-1]})$
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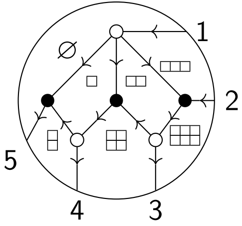
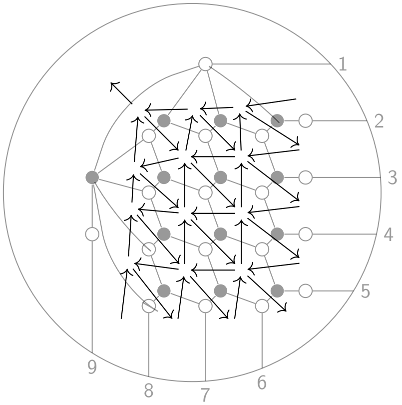
Theorem (Scott, Postnikov, Talaska, Müller-Speyer, RW)

A reduced plabic graph G with trip permutation $\pi_{k,n}$ gives rise to both an $\mathcal{A}$ and an $\mathcal{X}$ cluster structure on X.

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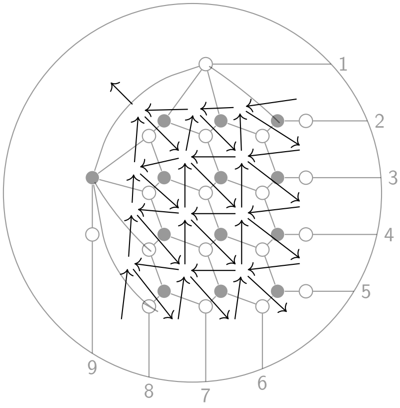

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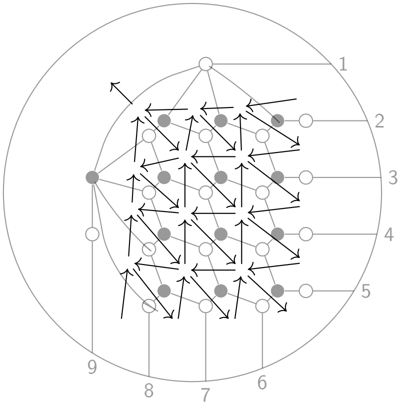

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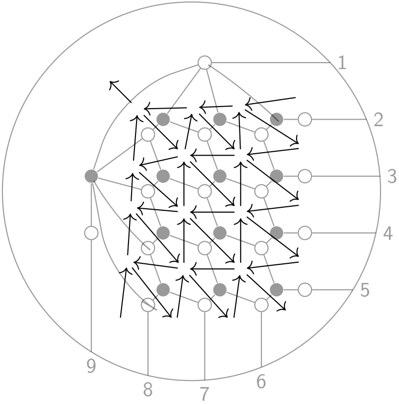

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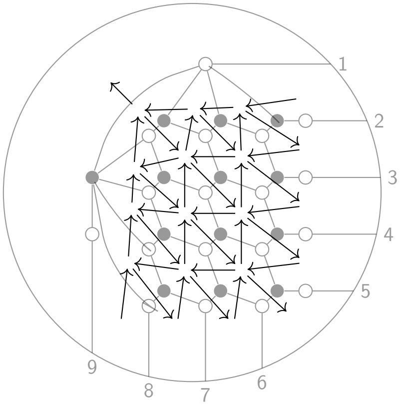

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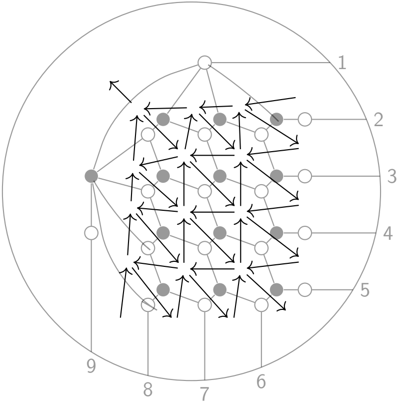

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$\rightsquigarrow$ can define a Newton-Okounkov body $\Delta_{\mathsf{val}_G}(D)$

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Lemma

Every choice of cluster torus gives rise to identifications

$$\mathcal{X}^{\text{trop}}(\mathbb{Z}) \equiv M \qquad\qquad \mathcal{A}^{\text{trop}}(\mathbb{Z}) \equiv N$$

Different identifications are related by piece-wise linear isomorphisms.

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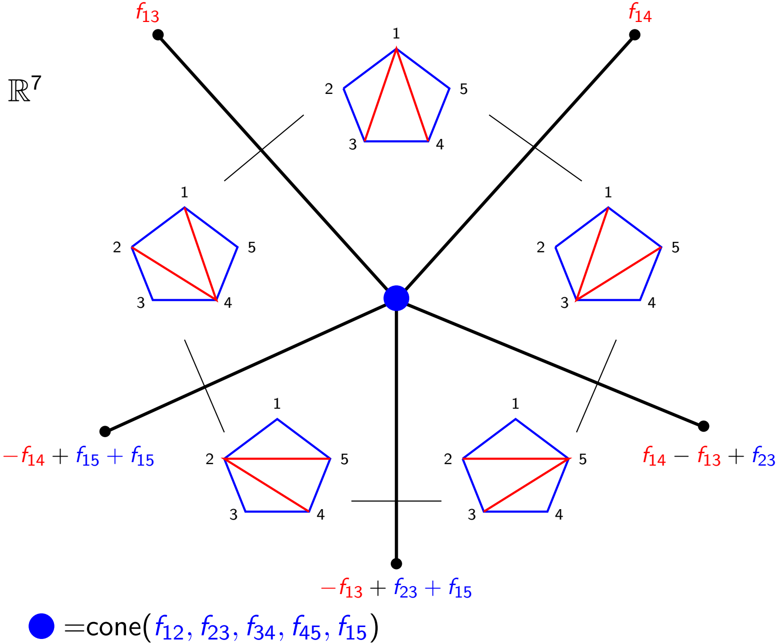
Theorem (GHKK)

Let $\mathcal{G}_v^+ := T_v^{-1}(\mathcal{C}_v^+)$. Then

$$\mathcal{G} = \bigcup_{v \in \mathbb{T}^r} \mathcal{G}_v$$

is a simplicilal fan in $M_{\mathbb{R}} := M \otimes \mathbb{R}$.


$\mathbb{R}^7$
f_{13}
f_{14}
$-f_{14} + f_{15} + f_{15}$
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- Consequences of the conjecture: An irreducible divisor D at infinity on a partial compactification of $\mathcal{V}$ gives rise to a function $f_D : \mathcal{V}^\vee \to \mathbb{C}$.
- Suppose $\mathcal{V} \subset Y$ is a snc compactification with anticanonical bounday $D = Y \setminus \mathcal{V}$ such that Ω has a pole at every irreducible component D_i of D.

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- If $\mathcal{V} = \mathcal{A}$ we let $\mathcal{V}^\vee = \mathcal{X}$ and vice-versa.
- Fock-Goncharov conjecture: $\mathcal{V}^{\text{trop}}(\mathbb{Z})$ parametrizes a $\mathbb{C}$-basis of the algebra of regular functions on $\mathcal{V}^\vee$.
- Consequences of the conjecture: An irreducible divisor D at infinity on a partial compactification of $\mathcal{V}$ gives rise to a function $f_D : \mathcal{V}^\vee \to \mathbb{C}$.
- Suppose $\mathcal{V} \subset Y$ is a snc compactification with anticanonical bounday $D = Y \setminus \mathcal{V}$ such that Ω has a pole at every irreducible component D_i of D. FG conjecture holds $\Rightarrow$ $W_D = \sum_i f_{D_i} : \mathcal{V}^\vee \to \mathbb{C}$.

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Example

Let $Y = \mathrm{Gr}_{n-k}(\mathbb{C}^n)$ and $\mathcal{X} = Y \setminus D_{\mathrm{ac}}$ the positroid variety inside Y. We have an associated mirror Landau-Ginzburg model

$$W : \mathcal{A} \to \mathbb{C}$$

$\mathcal{A}$ is the positrod variety inside $\mathrm{Gr}_k(\mathbb{C}^n)$ and $W = \sum_{i=1}^{n} W_i$, where

$$W_i := \frac{P_{[i+1,i+k-1]\cup\{i+k+1\}}}{p_{[i+1,i+k]}}$$

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Theorem (Marsh-Rietsch 13')

$$qH^*(Y)[q^{-1}] \cong Jac(W).$$

Theorem (RW 17')

The superpotential polytope associated to W and D is equal to the NO body associated to $\Delta_{\mathrm{val}_G}(D)$.

Scattering diagrams

Theorem (GHKK)

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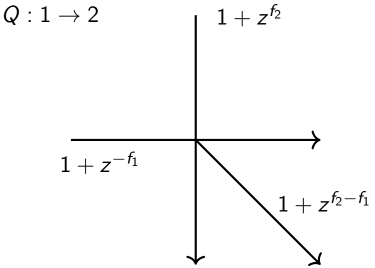
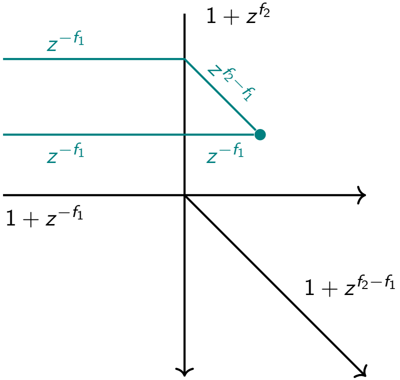


Theta functions on $\mathcal{A}$

Theorem (GHKK 14')

For each point $m \in \mathcal{X}^{\mathrm{trop}}(\mathbb{Z})$ consider the generating Laurent series $\vartheta_m \in \mathbb{C}[[T_N]]$ counting broken lines whose direction at infinity is m and whose endpoint is the positive orthant. If ϑ_m is a finite sum then it is a global function on $\mathcal{A}$.



Theta functions $\mathcal{X}$ via $\mathcal{A}^{\text{prin}}$

Definition

Let Q be a quiver. The $\mathcal{A}$-variety with principal coefficients $\mathcal{A}_Q^{\text{prin}}$ is the $\mathcal{A}$-variety associated to the principal extension Q^{prin}.

$$Q: \quad 1 \longrightarrow 2 \longrightarrow 3 \qquad\qquad Q^{\text{prin}}: \quad \begin{array}{ccccc} 1' & & 2' & & 3' \\ \uparrow & & \uparrow & & \uparrow \\ 1 & \longrightarrow & 2 & \longrightarrow & 3 \end{array}$$

- The lattice associated to $\mathcal{A}^{\text{prin}}$ is $M \times N$
- T_N acts on $\mathcal{A}^{\text{prin}} \rightsquigarrow$ quotient map $\mathcal{A}^{\text{prin}} \to \mathcal{X}$

$$\begin{array}{ccc} T_N & \hookrightarrow & T_N \times T_M \\ & & \downarrow \\ & & T_M \end{array} \qquad\qquad \begin{array}{ccc} T_N & \hookrightarrow & \mathcal{A}^{\text{prin}} \\ & & \downarrow \\ & & \mathcal{X} \end{array}$$

Theta functions $\mathcal{X}$ via $\mathcal{A}^{\text{prin}}$

Theorem (GHKK)

- For every point $(m, n) \in (\mathcal{X}^{\text{prin}})^{\text{trop}}(\mathbb{Z})$ the series $\vartheta_{(m,n)}$ has a well defined T_N-weight
- The tropical space $\mathcal{A}^{\text{trop}}$ identifies with the T_N-weight 0 slice
- Theta functions on $\mathcal{X}$ are the weight 0 ϑ-functions on $\mathcal{A}^{\text{prin}}$

Definition

Let $\mathcal{A} \subset \overline{\mathcal{A}}$ be a compactification given by setting frozen variables to 0. The ϑ-potential is

$$W_\vartheta = \sum_i \vartheta_i^{\mathcal{X}} : \mathcal{X} \to \mathbb{C},$$

ϑ_i is the ϑ-function associated to the i^{th} component of $\overline{\mathcal{A}} \setminus \mathcal{A}$.

For each cluster variety $\mathcal{A}_Q$ let $\Theta_{\mathcal{A}} \subset \mathcal{X}_Q^{\text{trop}}(\mathbb{Z})$ be the set of points that correspond to polynomial theta functions.

$$\text{mid}(\mathcal{A}) = \langle \vartheta_m \mid m \in \Theta_{\mathcal{A}} \rangle$$

Theorem (GHKK)

$\Theta_{\mathcal{A}}$ contains the $\mathbf{g}$-fan and $\{\vartheta_m \mid m \in \Theta_{\mathcal{A}}\}$ is a basis for $\text{mid}(\mathcal{A})$.

Theorem (Bossinger-Cheung-Magee-NC)

Let Q be arbitrary and fix an identification $\mathcal{X}_Q^{\text{trop}} \overset{s}{\equiv} M$. Then there exists a linear dominance order $\prec_s$ and a valuation

$$\mathbf{g}_s : \text{mid}(\mathcal{A}_Q) \to (M, \prec_s)$$

such that $\mathbf{g}_s(\vartheta_m) = m$ and the theta basis is an adapted basis.

For each cluster variety $\mathcal{X}$ let $\Theta_{\mathcal{X}} := \Theta_{\mathcal{A}^{\mathrm{prin}}} \bigcap \mathcal{A}^{\mathrm{trop}}(\mathbb{Z})$.

$$\mathrm{mid}(\mathcal{X}) = \langle \vartheta^{\mathcal{X}}_{(n,m)} \mid (n,m) \in \Theta_{\mathcal{X}} \rangle$$

Theorem (Bossinger-Cheung-Magee-NC)

Let Q be arbitrary and fix an identification $(\mathcal{X}^{\mathrm{prin}})^{\mathrm{trop}}(\mathbb{Z}) \overset{s}{\equiv} N \oplus M$. Then there exists a valuation

$$\mathbf{c}_s : \mathrm{mid}(\mathcal{A}_Q) \to (N, <_{\mathrm{lex}})$$

such that $\mathbf{c}_s(\vartheta^{\mathcal{X}}_{(n,m)}) = n$ and the theta basis is an adapted basis.

Recall, $X = \mathrm{Gr}_{n-k}(\mathbb{C}^n) \setminus D$.

- There are 2 potentials W and W_ϑ on $X^\vee$
- have valuations val_G and $\mathbf{c}_s$ on the section ring of $R(D) \oplus_{j\geq 1} \Gamma(X, \mathcal{O}(jD)) \subset \mathbb{C}(X)$.
- Have valuation $\mathbf{g}_s$ on $R(D^\vee) \underline{=\subset} \mathbb{C}(X^\vee)$
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Theorem (BCMNC)

1. $\mathrm{val}_G = \mathbf{c}_{s^{\mathrm{op}}}$
2. there is a unique cluster ensemble isomorphism $p : X^\vee_{\mathcal{A}} \to X^\vee_{\mathcal{X}}$ such that $p^*(\vartheta_i^{\mathcal{X}}) = W_i$
3. the dual cluster ensemble map $p^\vee$ gives identification $(p^\vee)^*(\Delta_G(D)) = \Delta_{\mathbf{g}_s}(D^\vee)$
4. the central fiber of the toric degeneration associated to val_G and the central fiber of the $\mathcal{A}^{\mathrm{prin}}$ toric degeneration are isomorphic.

Cluster ensemble maps

Every matrix of the form

$$\begin{bmatrix} B^Q_{m\times m} & B^Q_{m\times f} \\ B^Q_{f\times m} & *_{f\times f} \end{bmatrix}_{(m+f)\times(m+f)}$$

where $b_{ij} = \#\{\text{arrows } i \to j\} - \#\{\text{arrows } j \to i\}$, gives rise to a map cluster ensemble map

$$\mathcal{A}_Q \to \mathcal{X}_Q$$

Theorem (BCMNC)

There is a well defined notion of duality of cluster ensemble maps. The Euler from of the dimer algerba associated G gives rise to p.