2021-04-114: Wall-crossing for Newton-Okounkov bodies

ICERM Workshop Algebraic Geometry and Polyhedra

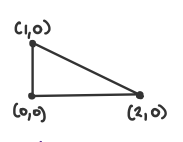
Wall Crossing for Newton-Okounkov bodies

joint work with Megumi Harada

<u>Outline</u>: ① Newton-Okounkov bodies
② Wall-Crossing for NO-bodies

The Newton polytope of $f = \sum_\alpha c_\alpha x^\alpha \in \mathbb{C}[x_1, \dots, x_n]$ is $\mathrm{Newt}(f) := \mathrm{conv}\{\alpha \mid c_\alpha \neq 0\}$.

$\mathrm{Newt}(3x_1^2 + x_2 - 1) =$



Bernstein-Khovanskii-Kuchnirenko Thm: $A \subseteq \mathbb{Z}^n$ finite, $L_A := \{\sum_{\alpha \in A} c_\alpha x^\alpha \mid c_\alpha \in \mathbb{C}\}$. For a generic choice of $f_1, \dots, f_n \in L_A$ the number of solutions to $f_1 = \dots = f_n = 0$ in $(\mathbb{C}^*)^n$ is $n!\,\mathrm{vol}(\mathrm{conv}(A))$.

Δ polytope $\rightsquigarrow X_\Delta \subseteq \mathbb{P}^s$ projective toric variety.

$n!\,\mathrm{vol}(\Delta) = \deg(X_\Delta)$

X, irreducible projective variety over $\mathbb{C}$

A, homogeneous coordinate ring of X

NO-bodies [Okounkov, Lazarsfeld-Mustaţă, Kaveh-Khovanskii]

valuation on $A \rightsquigarrow$ convex body Δ

$\deg(X) = n!\,\mathrm{vol}(\Delta)$

Thm [Anderson, 2013]: When Δ is a polytope of $\dim(\Delta) = \dim(X)$, there is a degeneration of X to X_Δ.

Thm [Harada-Kaveh, 2015]: When X is smooth there exists a full dimensional Hamiltonian torus action with moment map image Δ.

Cluster varieties

Equip $\mathbb{Z}^n$ w/ total order $\leq$.

A valuation is $\nu: A \setminus \{0\} \longrightarrow \mathbb{Z}^n$ st

① $\nu(f+g) \geq \max(\nu(f), \nu(g))$

② $\nu(fg) = \nu(f) + \nu(g)$

③ $\nu(c) = 0 \quad \forall c \in \mathbb{C}^*$

Example: $\nu(\sum c_\alpha x^\alpha) := \min\{\alpha \mid c_\alpha \neq 0\}$ is a valuation on $\mathbb{C}[x_1, \dots, x_n]$.

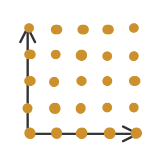


image of ν

Another example: X = hypersurface $y^2z - x^3 + 7xz^2 - 2z^3$

$$M = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -3 & 0 \end{bmatrix}$$

Valuation $\nu: \mathbb{C}[X] \longrightarrow \mathbb{Z}^2$

$$\sum c_{\alpha,\beta,\gamma} x^\alpha y^\beta z^\gamma \longmapsto \min\left\{ M \cdot \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \text{ st } c_{\alpha,\beta,\gamma} \neq 0 \right\}$$

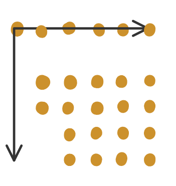
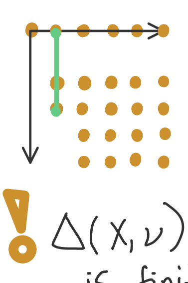


image of ν = semigroup generated by the columns of M

NO-body

$\Delta(X, \nu) = \overline{\mathrm{cone}(\mathrm{image}(\nu))} \cap \{x_1 = 1\}$



$\Delta(X, \nu) =$ convex hull of columns of M

! $\Delta(X, \nu)$ is a polytope if $\mathrm{image}(\nu)$ is finitely generated.

[Kaveh-Manon]: tropical geometry
$\Updownarrow$
valuations w/ polytopal NO-bodies

I ideal, $\mathrm{trop}(I) := \left\{ w \in \mathbb{Q}^n \,\middle|\, \begin{array}{l} \mathrm{in}_w I \text{ contains} \\ \text{no monomials} \end{array} \right\}$

w/ fan structure having cones

$C_w := \overline{\{w' \in \mathrm{trop}(I) \mid \mathrm{in}_{w'} I = \mathrm{in}_w I\}}$

Example: $\mathrm{trop}(\langle y^2z - x^3 + 7xz^2 - 2z^3 \rangle)$

$\mathrm{Cone}((0,1,0), \pm\mathbb{1})$ $\quad \mathrm{in}_w I = \langle -x^3 + 7xz^2 - 2z^3 \rangle$

$\mathrm{cone}((1,0,0), \pm\mathbb{1})$ $\quad \mathrm{in}_w I = \langle y^2z - 2z^3 \rangle$

$\mathrm{Cone}((-2,-3,0), \pm\mathbb{1})$ $\quad \mathrm{in}_w I = \langle y^2z - x^3 \rangle$

A cone C in $\mathrm{trop}(I)$ is prime if $\mathrm{in}_w I$ is prime for $w \in C^\circ$.

<u>Thm</u> [Kaveh-Manon]: Let C be a prime cone of $\mathrm{trop}(I)$.

① $\{u_1, \dots, u_r\} \subseteq C$ linearly ind, $r = \dim C$.

② M = mtx w/ rows $u_1, \dots, u_r$

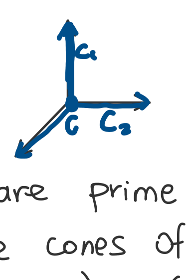
Construct a valuation ν_C st

$\Delta(X, \nu_C) =$ convex hull of the columns of M

Example: Grassmannian $Gr(2,4)$

$I_{2,4} = \langle P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} \rangle$

$\mathrm{trop}(I_{2,4}) =$



$\times \mathcal{L} \subseteq \mathbb{R}^6$

4D vector space.

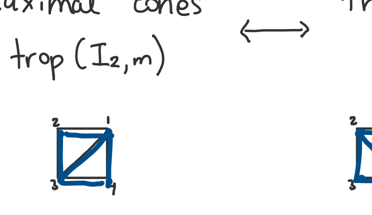
All cones are prime

C_1, C_2 prime cones of maximal dim.

$\rightsquigarrow \Delta(Gr(2,4), \nu_{C_1}), \Delta(Gr(2,4), \nu_{C_2})$.

Semigroups $\mathrm{image}(\nu_{C_1})$, $\mathrm{image}(\nu_{C_2})$

Maximal cones $\mathrm{trop}(I_{2,m})$ $\longleftrightarrow$ triangulations of labelled m-gon



$\Phi(z_{12}, z_{13}, z_{14}, z_{23}, z_{34}) = (z_{12}, z_{14}, z_{23}, z_{24}, z_{34})$

$z_{24} = \mathrm{trop}\left(\frac{z_{12}z_{34} + z_{14}z_{23}}{z_{13}}\right)$

[Nohara-Ueda]

Geometric wall-crossing for NO-bodies

X projective variety

$A = \mathbb{C}[x_1, \dots, x_n]/I$

C_1, C_2 prime cones of <u>$\mathrm{trop}(I)$</u> st

C_1, C_2 <u>maximal dimension</u>, and

C_1, C_2 <u>share a facet C.</u>

Then

$$\begin{tikzcd}[column sep=small] \Delta(X, \nu_{C_1}) \arrow[dr, "\pi"] & & \Delta(X, \nu_{C_2}) \arrow[dl, "\pi"'] \\ & \Delta(X, \nu_C) & \end{tikzcd}$$

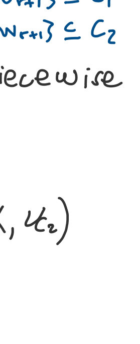
where $\pi: \mathbb{R}^d \longrightarrow \mathbb{R}^{d-1}$ projection

<u>Thm</u> [E.-Harada]: The fibers $\pi^{-1}(p) \cap \Delta(X, \nu_{C_1})$ and $\pi^{-1}(p) \cap \Delta(X, \nu_{C_2})$ are intervals of the same length.

$\Delta(X, \nu_{C_1})$

$\Delta(X, \nu_C)$

$\Delta(X, \nu_{C_2})$



① Change choice of $\{u_1, \dots, u_r\} \subseteq C$

Obtain linearly isomorphic polytope.

② $\{u_1, \dots, u_r\} \subseteq C = C_1 \cap C_2$, $\quad \{u_1, \dots, u_r, u_{r+1}\} \subseteq C_1$, $\{u_1, \dots, u_r, w_{r+1}\} \subseteq C_2$

<u>Thm</u> [E.-Harada]: Obtain 2 piecewise linear maps Φ_S, Φ_F

$$\begin{tikzcd}[column sep=small] \Delta(X, \nu_{C_1}) \arrow[rr, "\Phi_{S \text{ or } F}"] \arrow[dr, "\pi"'] & & \Delta(X, \nu_{C_2}) \arrow[dl, "\pi"] \\ & \Delta(X, \nu_C) & \end{tikzcd}$$

<u>Remarks</u>:

[Ilten-Manon]: geometric wall-crossing can be derived from the theory of complexity-1 T-varieties

[Ilten]: interpretation of geometric wall-crossing as a generalization of the combinatorial mutation of Akhtar-Coates-Galkin-Kasprzyk

[E.-Harada]: algebraic wall-crossing, i.e. bijection

$$\begin{tikzcd}[column sep=small] \mathrm{image}(\nu_{C_1}) \arrow[rr] \arrow[dr] & & \mathrm{image}(\nu_{C_2}). \arrow[dl] \\ & \mathrm{image}(\nu_C) & \end{tikzcd}$$

[E.-Harada, Bossinger-Mohammadi-Nájera Chávez]: For $Gr(2,m)$, Φ_F arises from cluster algebra structure.