# Combinatorics and real lifts of bitangents to tropical plane quartics

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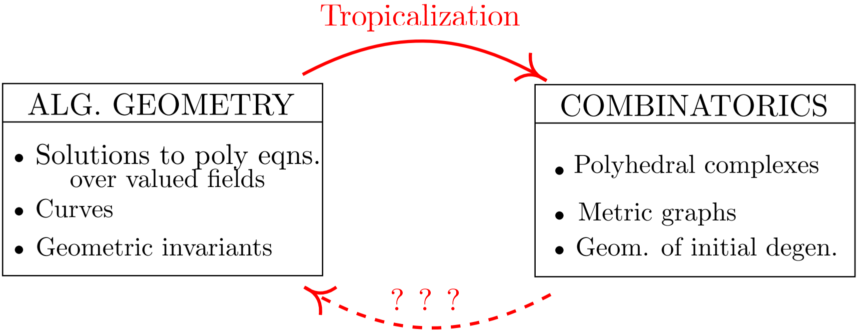
# Tropical Mathematics



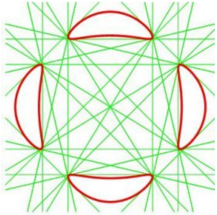

- Feature/Bug: Tropicalization is *very sensitive* to choice of embeddings.
- Tropicalization is not injective; often, see tropical superabundance.
- GOAL: Use combinatorics to study real bitangents to sm. plane quartics.

# Today's focus: two classical results in Algebraic Geometry

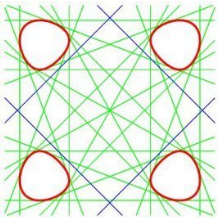
Plücker (1834): A sm. quartic curve in $\mathbb{P}^2_{\mathbb{C}}$ has exactly 28 bitangent lines.

Zeuthen (1873): 4, 8, 16 or 28 real bitangents (real curve: $\mathcal{V}_{\mathbb{R}}(f) \subset \mathbb{P}^2_{\mathbb{R}}$).

| The real curve | Real bitangents | The real curve | Real bitangents |
|---|---|---|---|
| 4 ovals | 28 | 1 oval | 4 |
| 3 ovals | 16 | 2 nested ovals | 4 |
| 2 non-nested ovals | 8 | empty curve | 4 |



**Trott:** 28 totally real bitangents.



**Salmon:** 28 real, 24 totally real.

ISSUE: Plücker's result fails tropically! But we can fix it.

GOAL: Use tropical geometry to find bitangents over $\mathbb{C}\{\{t\}\}$ and $\mathbb{R}\{\{t\}\}$.

# 28 bitangent lines to sm. plane quartics over $\mathbb{K} = \overline{\mathbb{C}((t))}$.

Plücker-Zeuthen: A sm. quartic curve in $\mathbb{P}^2_{\mathbb{K}}$ has exactly 28 bitangent lines (4, 8, 16 or 28 real bitangents, depending on topology of the real curve.)

- What happens tropically?

Baker-Len-Morrison-Pflueger-Ren (2016): Every tropical smooth quartic in $\mathbb{R}^2$ has infinitely many tropical bitangents (in **7 equivalence classes**.)
Conjecture [BLMPR]: Each bitangent class hides 4 classical bitangents.

- Three independent answers (with different approaches):

Chan-Jiradilok (2017): Conjecture holds for tropical $K_4$-curves.

Len-Jensen (2018): Each class *always* lifts to 4 classical bitangents.

Len-Markwig (2020): We have an **algorithm** to reconstruct the 4 classical bitangents $\ell = y + m + nx$ and the tangencies under mild genericity.

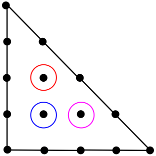
**Question 1:** What is a tropical bitangent line? Tropical tangencies?

**Question 2:** What is a tropical bitangent class?
**Answer:** Continuous translations preserving bitangency property.

# 28 bitangent lines to sm. plane quartics over $\mathbb{K} = \overline{\mathbb{C}((t))}$.

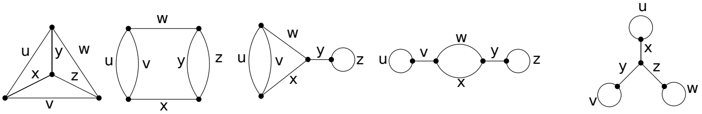
Theorem: There are 28 classical bitangents to sm. plane quartics over $\mathbb{K}$ **but** 7 tropical bitangent classes to their smooth tropicalizations in $\mathbb{R}^2$.

Trop. sm. quartic = dual to unimodular triangulation of $\Delta_2$ of side length 4.



$\rightsquigarrow$ duality gives a genus 3 planar metric graph.

Possible cases:



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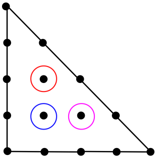
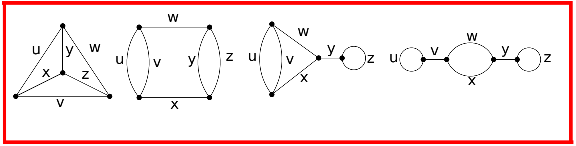
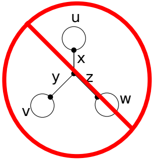


$\rightsquigarrow$ duality gives a genus 3 planar metric graph.

Possible cases: [BLMPR '16]



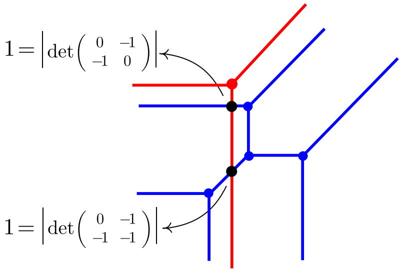

Brodsky-Joswig-Morrison-Sturmfels (2015): Newton subdivisions give linear restrictions on the lengths $u$, $v$, $w$, $x$, $y$, $z$ of the edges.

Hahn-Markwig-Ren-Tyomkin (2019): Higher-dimensional linear re-embeddings realize all five graphs and with no length restrictions.

# Basic facts about general tropical plane curves:

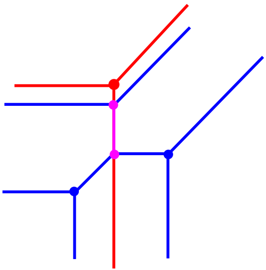
(1) Interpolation for *general* pts in $\mathbb{R}^2$ holds tropically (Mikhalkin's Corresp.)
(unique line through 2 gen. points, unique conic through 5 gen. points,...)

(2) *General* trop. plane curves intersect as expected (Trop. Bézout.)




Proper intersection at 2 pts

vs.



Non-proper intersection

# Basic facts about general tropical plane curves:

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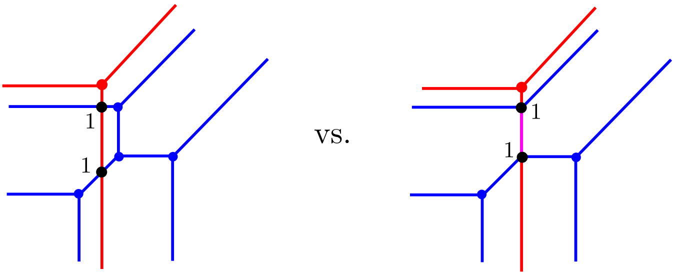


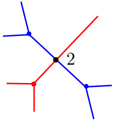

Proper intersection at 2 pts

Stable intersection at 2 pts

Non-proper case: Replace usual intersection with stable intersection.

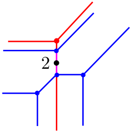
$$C_1 \cap_{st} C_2 := \lim_{\underline{\varepsilon} \to (0,0)} C_1 \cap (C_2 + \underline{\varepsilon}).$$

# Tropical bitangent lines to tropical smooth quartics in $\mathbb{R}^2$:




Proper tangency

vs.



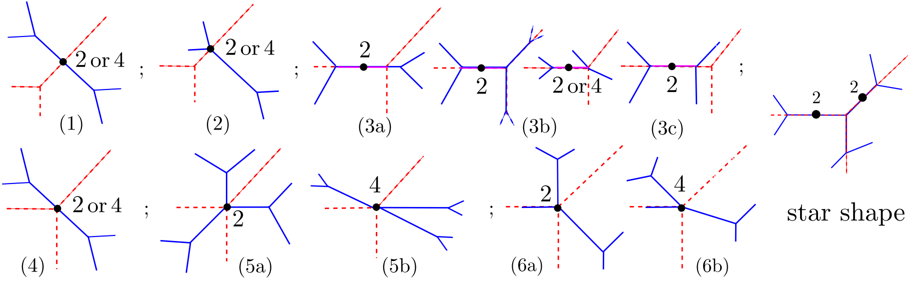

Midpoint tangency

Definition: $\Lambda =$  is a **bitangent line** to the quartic $\Gamma$ if and only if:

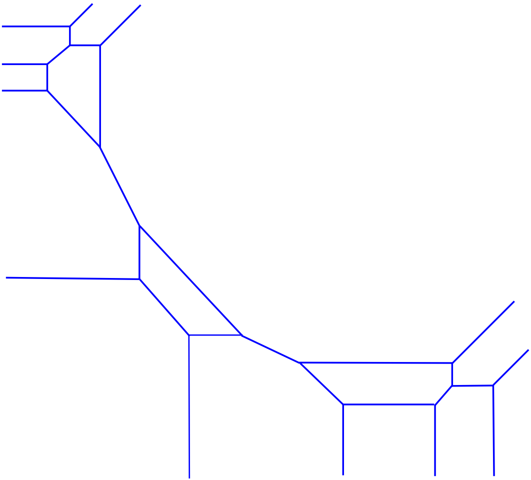
(i) $\Lambda \cap \Gamma$ has 2 conn. components of stable intersection mult. 2 each; or

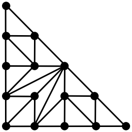
(ii) $\Lambda \cap \Gamma$ is connected and its stable intersection multiplicity is 4.

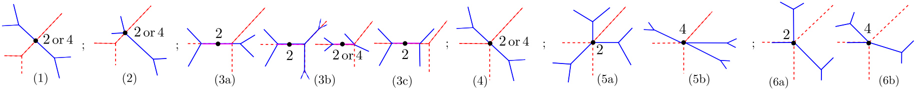
[L-M '20]: 6 local tangency types between $\Lambda$ and $\Gamma$ (up to $\mathbb{S}_3$-symmetry).



# 28 classical bitangents vs. 7 tropical bitangent classes.







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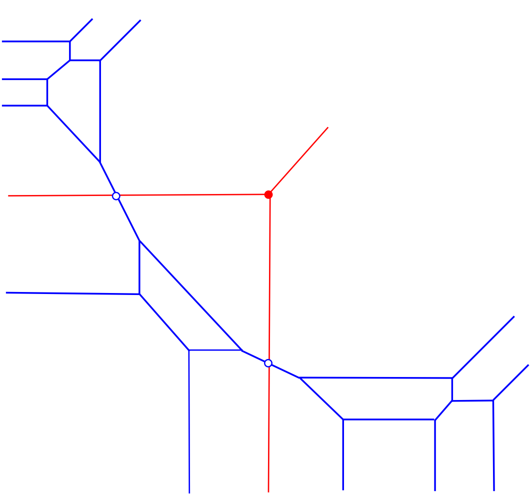

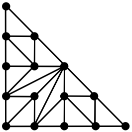

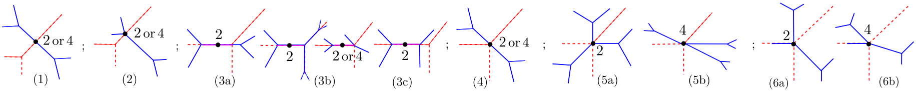

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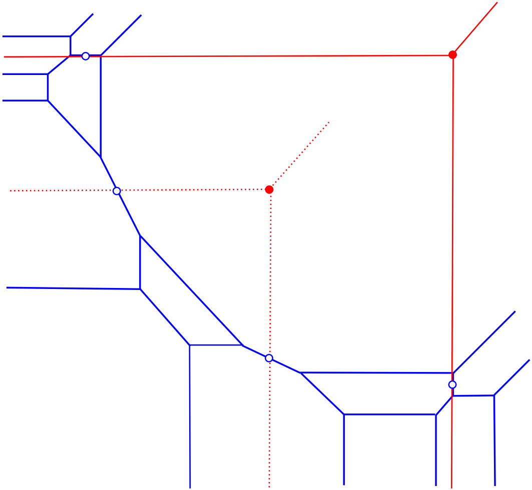

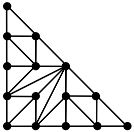

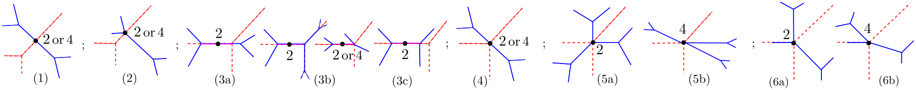

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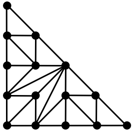

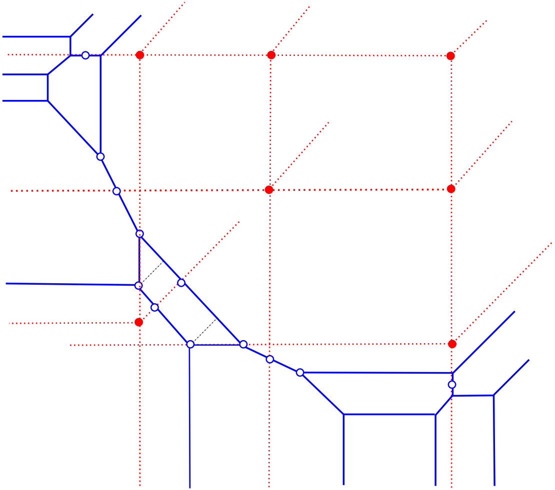

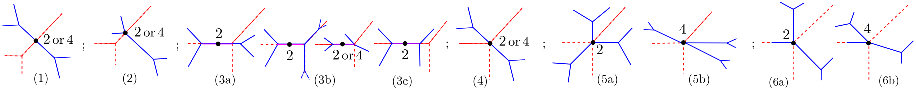

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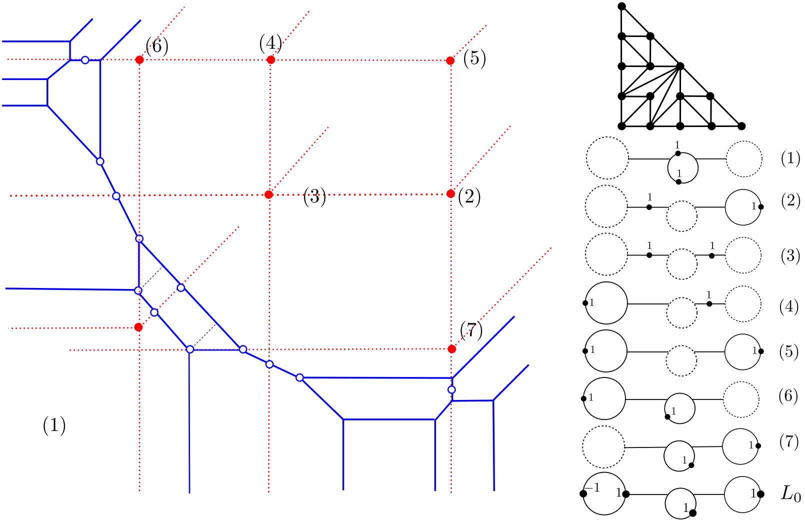
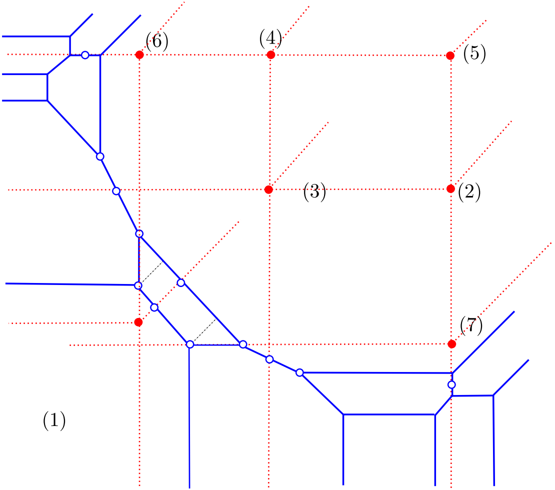
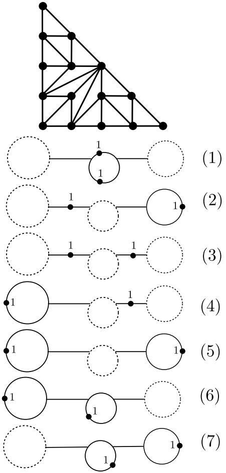



Zharkov (2010): Trop. theta characteristics $\theta_i$ on the metric graph $G$:

$$2\theta_i \sim K_G = \sum_{x \in G}(\mathrm{val}(x) - 2)x \qquad ; \qquad (\theta_i)_i \leftrightarrow H_1(G, \mathbb{Z}/2\mathbb{Z})$$

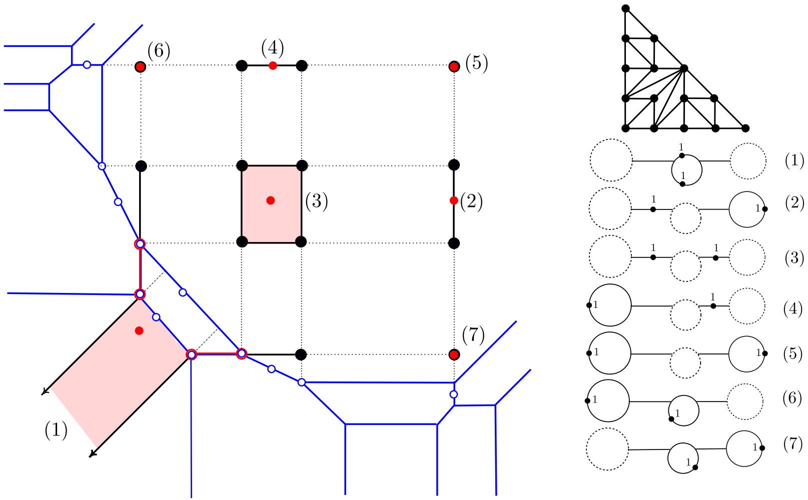
# 28 classical bitangents vs. 7 tropical bitangent classes.




[BLMPR '16]: 7 effective trop. theta characteristics on **skeleton** of tropical sm. quartic Γ in $\mathbb{R}^2$ produce 7 tropical bitangent lines Λ to Γ.

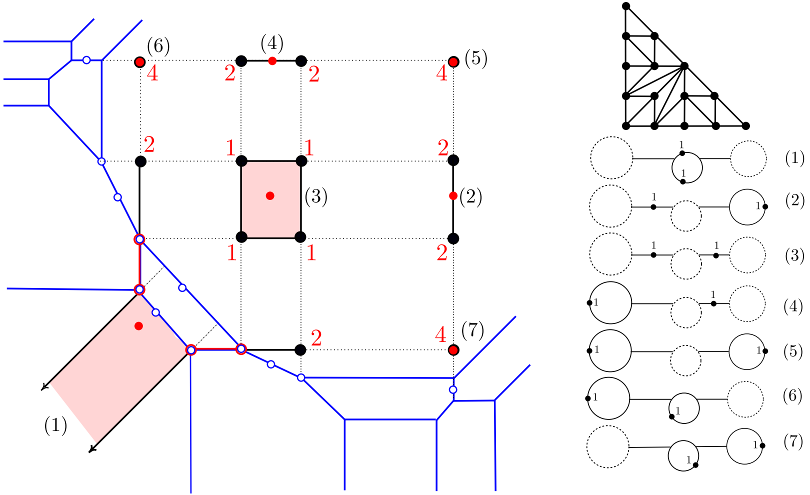
# 28 classical bitangents vs. 7 tropical bitangent classes.




[BLMPR '16]: Equiv. class = move Λ continuously, staying bitangent.

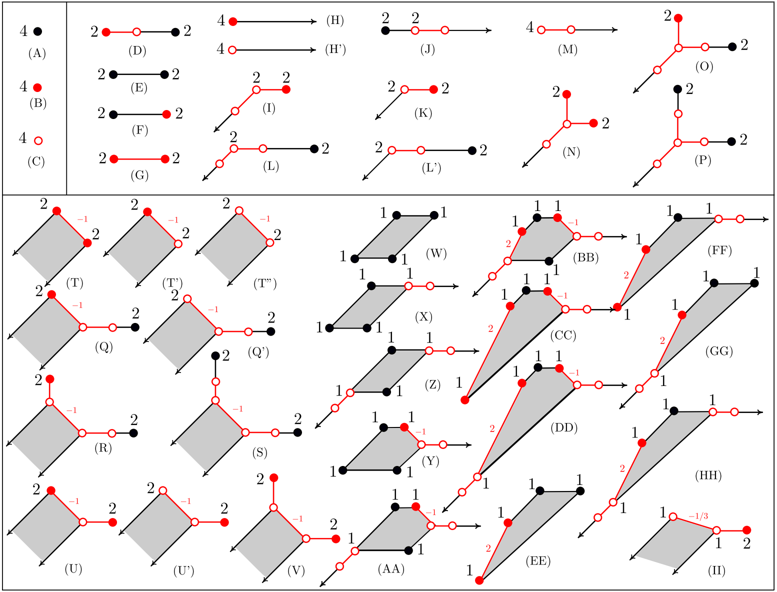
[L-J '18, L-M '20]: Each bitangent class lifts to 4 classical bitangents.

# 28 classical bitangents vs. 7 tropical bitangent classes.




C.-Markwig (2020): There are **41 shapes** of bitangent classes (up to symm.) They are **min-tropical** convex sets. Liftings come from vertices.
**Over** $\mathbb{R}$**:** liftings on each class are either all (totally) real or none is real.

# THM 1: Classification into 41 bitangent classes (up to $\mathbb{S}_3$-symmetry)

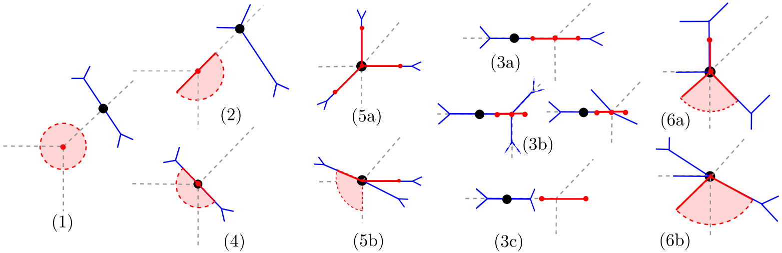



Bitangent line $\longleftrightarrow$ location of its vertex.

# Proof sketch of Combinatorial classification Theorem

**Step 1:** Identify edge directions for Γ involved in local tangencies.

**Step 2:** Identify local moves of the vertex of Λ that preserve one tangency



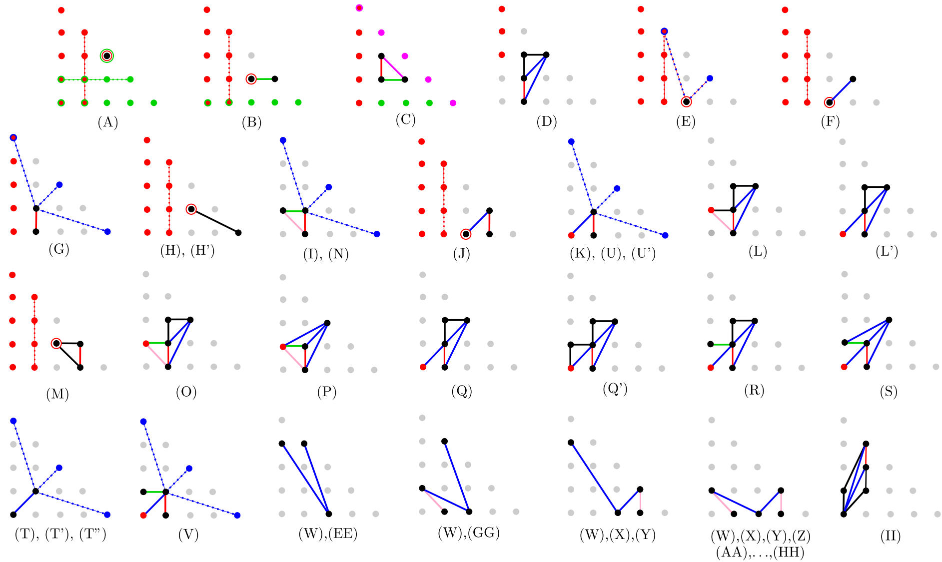

**Step 3:** Interpret tangency types from cells in the Newton subdivision.

**Step 4:** Classify the shapes using 3 properties of its members:

| max. mult. | proper | min. conn. comp. | shapes |
|---|---|---|---|
| 4 | yes | 1 | (II) |
| 4 | no | 1 | (C),(D),(L),(L'),(O),(P),(Q),(Q'),(R),(S) |
| 2 | yes/no | 2 | rest |

For the last row, refine using dimension and boundedness of its top cell.

# Corollary: Partial Newton subdivisions for all 41 bitangent shapes.



# Lifting tropical bitangents to classical bitangents to $\mathcal{V}(q)$

Fix $\mathbb{K} = \mathbb{C}\{\{t\}\}$ (**complex** Puiseux series), $\mathbb{K}_{\mathbb{R}} = \mathbb{R}\{\{t\}\}$ (**real** P. s.)

- If $a = a_0 t^{\alpha} + h.o.t. \in \mathbb{K}$, write $\bar{a} := a_0 = \overline{a\, t^{-\alpha}}$ in $\mathbb{C}$ **(initial term)**.
- Assume bitangent line $\ell$ to $\mathcal{V}(q)$ is not vertical and all tangencies are in torus. Write $\ell\colon y + m + n\,x = 0$ with $m, n \in \mathbb{K}^*$.
- Set $\Lambda := \operatorname{Trop} \ell$ and $\Gamma := \operatorname{Trop} \mathcal{V}(q)$.

**Question:** When is $\ell$ tangent to $\mathcal{V}(q)$ at $p \in (\mathbb{K}^*)^2$?

**Answer:** $p$ satisfies $\ell = q = W = 0$, where $W = J(\ell, q)$ is the **Wronskian**.

Key Prop. [L-M '20]: If $p = (b_0\, t^{\alpha_0} + h.o.t, b_1\, t^{\alpha_1} + h.o.t)$, then

(i) $(\alpha_0, \alpha_1)$ is a **trop. tangency pt.** for $\Lambda$ and $\Gamma$.

(ii) The initial degenerations $\bar{q}, \bar{\ell}, \bar{W}$ from **lowest valuation terms** of $q, \ell, W$ locally at $p$ **vanish** at the initial term $\bar{p} := (b_0, b_1)$.

Thm. [L-M '20]: We can use $\bar{q} = \bar{\ell} = \bar{W} = 0$ to find $(\bar{m}, \bar{n}, \bar{p}) \in (\mathbb{C}^*)^4$.

# Lifting tropical bitangents to classical bitangents (cont)

$(\bar{m}, \bar{n}, \bar{p})$ and $\bar{q} = \bar{\ell} = \bar{W} = 0$ $\overset{???}{\rightsquigarrow}$ $(m, n, p)$ and $q = \ell = W = 0$

Multivariate Hensel's Lemma: If $J_{x,y,\bar{m}}(\bar{q}, \bar{\ell}, \bar{W})_{|\bar{p}} \neq 0$, then $(\bar{m}, \bar{p})$ lifts to a **unique solution** $(m, p)$. (Then, get $n$ from $\ell(p) = 0$.)

[L-M '20]: Analyzed local mult. 2 tangencies and saw:

(i) Tangencies in 2 ends of $\Lambda$ give complementary data ($\bar{m}, \bar{n}$ or $\bar{m}/\bar{n}$).

(ii) Tangencies in same end of $\Lambda$ with $\Lambda \cap \Gamma$ disconnected give non-compatible local equations (**genericity condition**.)

[L-M'20, C-M'20]: If mult. four, no hyperflexes:

| type | star | (5b) | (6b) |
|---|---|---|---|
| mult. | $2 \cdot 2$ | 1 | 1 |

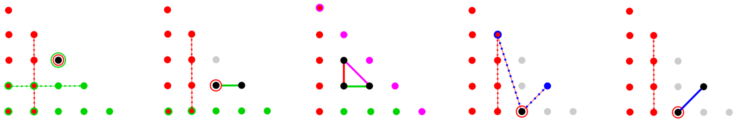
Thm.[L-M'20]: Local solns. for mult 1 in $\mathbb{Q}(\overline{a_{ij}})$ **but** otherwise in $\mathbb{Q}(\sqrt{\overline{a_{ij}}})$.

Crucial Obs.: Lifting lies in $\mathbb{K}_{\mathbb{R}}$ iff $(\bar{m}, \bar{n}, \bar{p}) \in \mathbb{R}^4$ and $q(x, y) \in \mathbb{K}_{\mathbb{R}}[x, y]$.

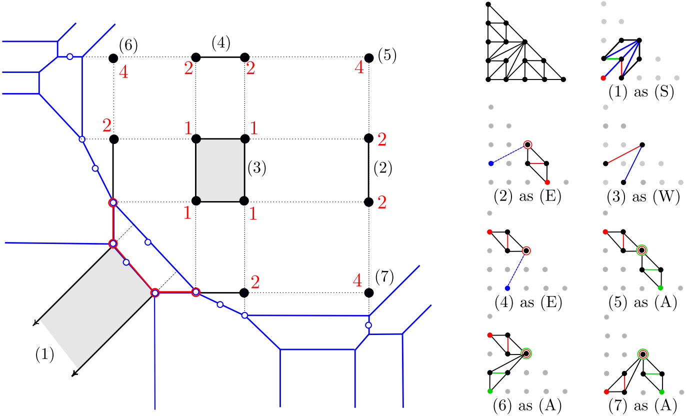
# THM 2: Real lifting sign conditions per representing bitangent class:

| Shape | Lifting conditions |
|---|---|
| (A) | $(-s_{1v}s_{1,v+1})^i s_{0i}s_{22} > 0$ and $(-s_{u1}s_{u+1,1})^j s_{j0}s_{22} > 0$ |
| (B) | $(-s_{1v}s_{1,v+1})^{i+1} s_{0i}s_{21} > 0$ and $(-s_{21})^{j+1} s_{31}{}^j s_{1v}s_{1,v+1}s_{j0} > 0$ |
| (C) | $\begin{cases} (-s_{11}s_{12})^i s_{0i}s_{20} > 0 \text{ and } (-s_{21}s_{12})^k s_{k,4-k}s_{20} > 0 & \text{if } j=2, \\ (-s_{11})^{i+1} s_{12}^i s_{21}s_{0i}s_{j0} > 0 \text{ and } (-s_{21})^{k+1} s_{12}^k s_{11}s_{k,4-k}s_{j0} > 0 & \text{if } j=1,3. \end{cases}$ |
| (H),(H') | $(-s_{1v}s_{1,v+1})^{i+1} s_{0i}s_{21} > 0$ and $s_{1v}s_{1,v+1}s_{21}s_{40} < 0$ |
| (M) | $(-s_{1v}s_{1,v+1})^{i+1} s_{0i}s_{21} > 0$ and $s_{1v}s_{1,v+1}s_{30}s_{31} > 0$ |
| (D) | $(-s_{10}s_{11})^i s_{0i}s_{22} > 0$ |
| (E),(F),(J) | $(-s_{1v}s_{1,v+1})^i s_{0i}\, s_{20} > 0$ |
| (G) | $(-s_{10}s_{11})^i s_{0i}\, s_{k,4-k} > 0$ |
| (I),(N) | $s_{10}s_{11}s_{01}s_{k,4-k} < 0$ |
| (K),(T),(U),(U'),(V) | $s_{00}s_{k,4-k} > 0$ |
| (L),(O),(P) | $s_{10}s_{11}s_{01}s_{22} < 0$ |
| (L'),(Q),(Q'),(R),(S) | $s_{00}s_{22} > 0$ |
| rest | no conditions |

**Indices:** relevant vertices in the Newton subdivision for each tangency, e.g.



(A) (B) (C) (E) (F)




Lifting conditions over the reals: **(1):** $s_{00}s_{22} > 0$; **(2):** $(-s_{21}s_{31})^3 s_{30}s_{22} > 0$
**(3):** none; **(4):** ; $(-s_{12}s_{13})^3 s_{03}s_{22} > 0$; **(5):** $(-s_{12}s_{13})^3 s_{03}s_{22} > 0, (-s_{21}s_{31})^3 s_{30}s_{22} > 0$;
**(6):** $-s_{12}s_{13}s_{03}s_{22} > 0, (-s_{01}s_{11})^0 s_{00}s_{22} > 0$; **(7):** $(-s_{10}s_{11})^0 s_{00}s_{22} > 0, -s_{21}s_{31}s_{30}s_{22} > 0$;

| Negative signs | Real bitangent classes | Number or real lifts | Topology |
|---|---|---|---|
| — | (1) and (3) | 8 | 2 non-nested ovals |
| $s_{31}$ | (1), (2), (3) and (7) | 16 | 3 ovals |
| $s_{13}$, $s_{31}$ | (1), . . . ,(7) | 28 | 4 ovals |
| $s_{13}$, $s_{31}$, $s_{22}$ | (3) | 4 | 1 oval |