Cyclotomic generating functions

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Outline

- Real-Rooted Generating Functions
- Cyclotomic Generating Functions (CGF's)
- Examples:
  - enumerative combinatorics
  - representation theory
  - geometry/algebra
- Properties:
  - asymptotics
  - γ-expansions
- Further directions

Generating Functions

Notation Let

$$X = \text{a set}$$

$$\text{stat}: X \to \mathbb{Z}_{\geq 0}.$$

Then

$$X^{\text{stat}}(q) = \sum_{x \in X} q^{\text{stat}(x)} \in \mathbb{Z}_{\geq 0}[q]$$

is a combinatorial generating function.

Ex (Subset sums)

$$X = \binom{[n]}{k} = \left\{ \begin{matrix} k\text{-element} \\ \text{subsets} \\ \text{of } \{1, \dots, n\} \end{matrix} \right\}$$

$\text{stat} = \text{sum}$ where $\text{sum}(S) = \sum_{s \in S} s$

$$\Rightarrow \binom{[3]}{2}^{\text{sum}}(q) = q^{1+2} + q^{1+3} + q^{2+3}$$

$$= q^3 + q^4 + q^5$$

Real-Rooted Generating Functions

Rem The location of roots of generating functions heavily influences the distribution of their coefficients.

Thm (Newton/Classical) Let $a_0 + a_1 q + \cdots + \overset{\neq 0}{\overset{\downarrow}{a_n}} q^n \in \mathbb{R}[q]$ have all real roots. Then $(a_0, a_1, \ldots, a_n)$ is ultra log-concave, meaning

$$\frac{a_{k-1}}{\binom{n}{k-1}} \frac{a_{k+1}}{\binom{n}{k+1}} \leq \left(\frac{a_k}{\binom{n}{k}} \right)^2 .$$

Cor If $a_k \geq 0$ $\forall k$, this sequence is log-concave and unimodal with no internal zeros.

Real-Rooted Generating Functions

<u>Thm</u> (Bender '73/Classical) Let $X^{(1)}, X^{(2)}, \ldots$ be a sequence of discrete random variables with real-rooted probability generating functions

$$E\left[q^{X^{(N)}}\right] = \sum_k \Pr\left(X^{(N)} = k\right) q^k \in \mathbb{R}_{\geq 0}[q].$$

If $\sigma^{(N)} \to \infty$, then $X^{(1)}, X^{(2)}, \ldots$ is asymptotically normal, meaning for all $t \in \mathbb{R}$,

$$\lim_{N \to \infty} \Pr\left(\frac{X^{(N)} - \mu^{(N)}}{\sigma^{(N)}} \leq t\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left(-\frac{x^2}{2}\right) dx.$$

Real-Rooted Generating Functions

$\boxed{\text{Ex}}$ (Stirling numbers of the second kind)

$X = \{\text{set partitions of } [n]\}$

$\text{stat} = \#\text{blocks}$

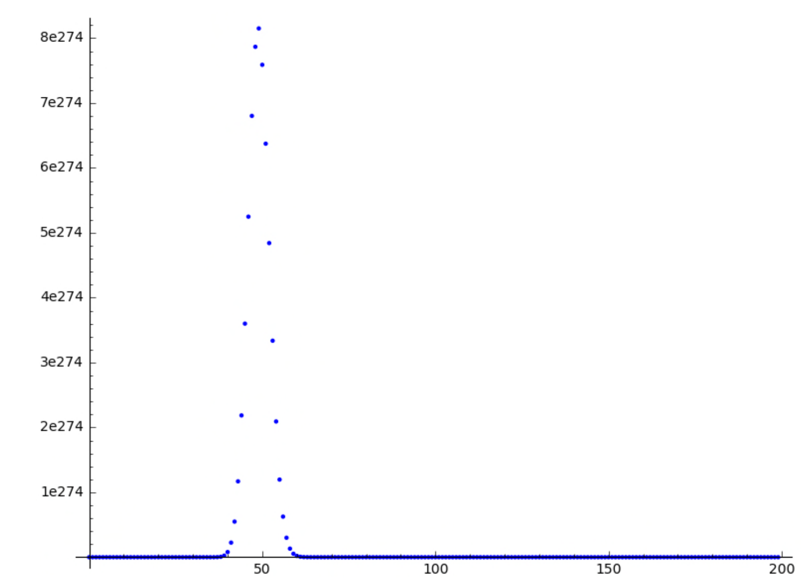
$$X^{\text{stat}}(q) = \sum_{k=1}^{n} S(n,k)\, x^k \leftarrow \text{real-rooted (Harper '67)}$$

$\chi^{(n)} = \#\text{blocks}$, uniform measure

$$E\left[q^{\chi^{(n)}}\right] = \frac{1}{B(n)} \sum_{k=0}^{n} S(n,k)\, x^k$$

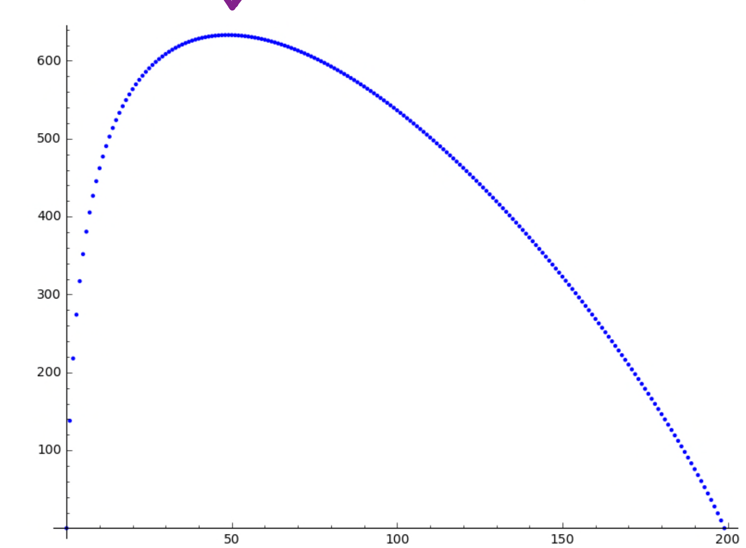
$(S(n,0), S(n,1), \ldots, S(n,n))$

$\Rightarrow$ is asymptotically normal as $n \to \infty$



n=200, S(n,k)

n=200, log S(n,k)



Cyclotomic Generating Functions

$\boxed{\text{Def}}$ (Billey–S. '21+) A polynomial $f(q) \in \mathbb{Z}_{\geq 0}[q]$ is a cyclotomic generating function (CGF) if it satisfies any of the following equivalent conditions.

(i) (Complex Form) The complex roots of $f(q)$ are all either roots of unity or 0.

(ii) (Cyclotomic Form) $f(q) = \alpha \cdot q^{\beta} \cdot \prod_j \Phi_{d_j}(q)$ where $\Phi_d(q) = \prod_{\substack{j=1 \\ \gcd(j,d)=1}}^{d} \left(q - \exp\left(2\pi i \frac{j}{d}\right)\right)$

(iii) (Rational Form) $f(q) = \alpha \cdot q^{\beta} \cdot \prod_j \frac{[a_j]_q}{[b_j]_q}$ where $[a]_q = \frac{1-q^a}{1-q} = 1 + q + \cdots + q^{a-1}$ is a q-integer

Cyclotomic Generating Functions

Why?

1) **Lemma** (Kronecker) Suppose $p(q) \in \mathbb{Z}[q]$ is monic and all complex roots are of norm ≤ 1. Then the roots of p are all either roots of unity or 0.

2) They show up a lot in practice, especially around root systems!

Next: examples!

Example: Enumerative Combinatorics

$\boxed{\text{Ex}}$ (q-binomials)

$$\binom{[n]}{k}^{\text{sum}}(q) = q^{\binom{k+1}{2}}\binom{n}{k}_q \quad \text{where} \quad \binom{n}{k}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}$$

(GF's!

$\boxed{\text{Ex}}$ (q-Factorials — inversion number/major index)

$$S_n^{\text{inv}}(q) = S_n^{\text{maj}}(q) = [n]_q! \quad \text{where} \quad [n]_q! = [n]_q[n-1]_q\cdots[1]_q$$

Example: Enumerative Combinatorics

$\boxed{\text{Ex}}$ (Iwahori–Matsumoto '65, Stembridge–Waugh '98, Zabrocki '03; see [BKS20a, §3.1])

For $\sigma \in S_n$, let $\mathrm{baj}(\sigma) = \sum_{i \in \mathrm{Des}(\sigma)} i(n-i)$.

Then
$$S_n^{\mathrm{baj-inv}}(q) = n\,[n-1]_q\,[n-2]_{q^2} \cdots [1]_{q^{n-1}} = n \prod_{i=1}^{n-1} [n-i]_{q^i}$$
$$= \prod_{i=1}^{n-1} \frac{[i(n-i)]_q}{[i]_q}$$

CGF!

Generalizes uniformly to finite Weyl groups

$\boxed{\text{Conj}}$ (J. Galovich) The coefficients of $S_n^{\mathrm{baj-inv}}(q)$ are unimodal.

Example: Representation Theory

$\boxed{\text{Ex}}$ (q-Weyl dimension formula; see Stembridge '94, §2.2-2.3)

Let $\mathfrak{g}$ = semisimple Lie algebra over $\mathbb{C}$

Φ = root system for $\mathfrak{g}$

Φ^+ = choice of positive roots

$\alpha_1, \ldots, \alpha_n$ = corresponding simple roots

$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$

Λ = weight lattice for Φ $\longrightarrow$

V_λ = irreducible $\mathfrak{g}$-module with highest weight λ

B_λ = a weight space basis for V_λ (i.e. respecting weight space decomposition)

Example: Representation Theory

Fact Partially order Λ by $\lambda \geq \mu \Leftrightarrow \lambda - \mu \in \mathrm{Span}_{\mathbb{Z}_{\geq 0}}\{\alpha_1, \ldots, \alpha_n\}$.
Let Λ_λ be the subposet of Λ consisting of V_λ's weights.
Then $(\Lambda_\lambda, \geq)$ is $\underbrace{\text{ranked}}_{r(\mu) = r(\nu)+1 \text{ when } \mu \text{ covers } \nu}$ by $r(\mu) = \langle \mu, \rho^\vee \rangle \in \frac{1}{2}\mathbb{Z}$ (equivalently, $r(\alpha_i) = 1$).

Q1 What is $\dim V_\lambda$ $(= |B_\lambda|)$?

Q2 (Refinement) What is the rank generating function of $(B_\lambda, \geq)$?

Example: Representation Theory

Thm (q-Weyl dimension formula)

Let V_λ be an irreducible $\mathfrak{g}$-module with highest weight λ.

Then the rank generating function of any weight space basis B_λ is

$$\sum_{b \in B_\lambda} q^{r(b)} = \sum_{\mu \in \Lambda} \dim (V_\lambda)_\mu \cdot q^{r(\mu)}$$

$$= \boxed{q^{-\langle \lambda, \rho^\vee \rangle} \prod_{\alpha \in \Phi^+} \frac{1-q^{\langle \lambda+\rho, \alpha^\vee \rangle}}{1-q^{\langle \rho, \alpha^\vee \rangle}}}$$

a CGF! (perhaps after shifting by $q^{1/2}$)

Example: Representation Theory

$\boxed{\text{Ex}}$ (Type A case)

$$s_\lambda(1,q,q^2,\dots,q^m) = q^{\sum_{i\geq 1}(i-1)\lambda_i} \prod_{1\leq i<j\leq m} \frac{[\lambda_i-\lambda_j+j-i]_q}{[j-i]_q}$$

Fact: unimodal

$\boxed{\text{Thm}}$ (Billey-S. '20+) These coefficients are asymptotically normal if the m parts of λ are distinct and $m\to\infty$.

In other regimes, they are asymptotically Irwin-Hall or generalized uniform sum distributions.

Example: Representation Theory

$\boxed{\text{Ex}}$ (Stanley's q-hook length formula)

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = q^{b(\lambda)} \frac{[n]_q!}{\prod_{c \in \lambda} [h_c]_q}$$

Not necessarily unimodal; see [BKS20a, Conj. 8.1] for characterization

$\boxed{\text{Conj}}$ [BKS20b, Conj. 9.1] The CGF's $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ are parity-unimodal, meaning their even- and odd-indexed coefficients are separately unimodal.

Example: Geometry

Ex The cohomology of the complete flag manifold is

$$\frac{\mathbb{Q}[x_1,\ldots,x_n]}{\langle e_1,\ldots,e_n\rangle}$$

with Hilbert series $[n]_q!$. This is unimodal by the Hard Lefschetz Theorem.

The Hilbert series of the λ-isotypic component is $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$.

Q Can geometry shed light on... isotypic parity-unimodality?

baj-inv unimodality?

internal zeros?

Example: Commutative Algebra

Def | Let $B = k[x_1, \dots, x_m]$ be a polynomial ring with grading $\deg(x_i) \in \mathbb{Z}_{\geq 1}$.
Let $\theta_1, \dots, \theta_m \in B$ be homogeneous polynomials. Set $A = k[\theta_1, \dots, \theta_m]$.
The sequence $\theta_1, \dots, \theta_m$ is a homogeneous system of parameters (HSOP) if B is a finitely generated A-module.

Example: Commutative Algebra

Thm (Macaulay; see Stanley '78, Cor. 3.2–3.3)
Let $\theta_1, \ldots, \theta_m$ be an HSOP in $k[x_1, \ldots, x_m]$ where $a_i = \deg(\theta_i)$ and $b_i = \deg(x_i)$.
Then $R = k[x_1, \ldots, x_m]/(\theta_1, \ldots, \theta_m)$ is a finite-dimensional k-vector space, and its Hilbert series is the cyclotomic generating function

$$\mathrm{Hilb}(R; q) = \sum_{j \geq 0} (\dim R_j)\, q^j = \prod_{i=1}^{m} \frac{[a_i]_q}{[b_i]_q} \in \mathbb{Z}_{\geq 0}[q].$$

Example: Commutative Algebra

Lemma If $a_1 \le \cdots \le a_m$ and $b_1 \le \cdots \le b_m$ arise from an HSOP, then $a_i \ge b_i$ $\forall i$.

Ex
$$\frac{[2]_q[3]_q[3]_q[8]_q[12]_q}{[1]_q[1]_q[4]_q[4]_q[6]_q} = 1 + 2q + 2q^2 + 2q^5 + 4q^6 + 2q^7 + 2q^{10} + 2q^{11} + q^{12} \in \mathbb{Z}_{\ge 0}[q]$$

is a CGF that <u>cannot</u> arise from an HSOP.

Q If $\prod_{i=1}^{m} \frac{[a_i]_q}{[b_i]_q} \in \mathbb{Z}_{\ge 0}[q]$, does $a_1 \le \cdots \le a_m$ majorize $b_1 \le \cdots \le b_m$ in the sense that $\sum_{i=1}^{l} a_i \ge \sum_{i=1}^{l} b_i$ and $\sum_{i=l}^{m} a_i \ge \sum_{i=l}^{m} b_i$?

CGF Asymptotics

Thm (Billey–S. '21+) Let $P_1(q), P_2(q), \ldots$ be a sequence of cyclotomic generating functions with $P_N(q) = q^{\beta_N} \prod_{a \in \mathbf{a}^{(N)}} [a]_q / \prod_{b \in \mathbf{b}^{(N)}} [b]_q$. Let $\mathcal{X}_1, \mathcal{X}_2, \ldots$ be a corresponding sequence of random variables with $E[q^{\mathcal{X}_N}] = P_N(q)/P_N(1)$.

IF

$$\limsup_{N \to \infty} \frac{\sum_{b \in \mathbf{b}^{(N)}} (b^2 - 1)}{\sum_{a \in \mathbf{a}^{(N)}} (a^2 - 1)} < 1$$

and

$$\lim_{N \to \infty} \frac{1}{(\max \mathbf{a}^{(N)})^2} \sum_{a \in \mathbf{a}^{(N)}} (a^2 - 1) = \infty$$

"generic case"

then $\mathcal{X}_1, \mathcal{X}_2, \ldots$ is asymptotically normal.

CGF Gamma Expansions

Def Given a symmetric polynomial of degree n, its gamma expansion is the sequence of its coefficients in the "centered binomial" basis $\{q^k(1+q)^{n-2k}\}$.

Ex

$$\frac{[8]_q!}{[6]_q[5]_q[3]_q[2]_q^3} = 1+q+3q^2+3q^3+5q^4+5q^5+7q^6+6q^7+7q^8+5q^9+5q^{10}+3q^{11}+3q^{12}+q^{13}+q^{14}$$

$$= 1\cdot q^0(1+q)^{14}+(-13)\cdot q^1(1+q)^{12}+68\cdot q^2(1+q)^{10}+(-183)\cdot q^3(1+q)^8$$
$$+268\cdot q^4(1+q)^6+(-206)\cdot q^5(1+q)^4+72\cdot q^6(1+q)^2+(-8)\cdot q^7(1+q)$$

$$\Rightarrow (\gamma_0,\gamma_1,\ldots)=(1,-13,68,-183,268,-206,72,-8)$$

CGF Gamma Expansions

Prop The absolute value of the coefficients of the γ-expansion of any cyclotomic generating function are unimodal.

Pf Uses Lucas atoms of Sagan-Tirrell '19+ and work of Bennett-Carillo-Machacek-Sagan '20.

Further Directions

Q | How many monic CGF's are there of degree n?

Warmup | How many monic polynomials in $\mathbb{Z}[q]$ of degree n have unit roots?

Q | What are the minimal generators of the monoid of monic CGF's? What about the unimodal or log-concave submonoids? How do the volumes of these cones compare?

Q | Are there unimodal CGF's whose coefficients are not asymptotically DUSTPAN distributions? (See Billey-S. 20+)

References

- [BKS20a] Billey-Konvalinka-Swanson
  "Asymptotic normality of the major index on standard tableaux"
  Adv. in Appl. Math. 113 (2020)
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- Billey-Swanson '20+
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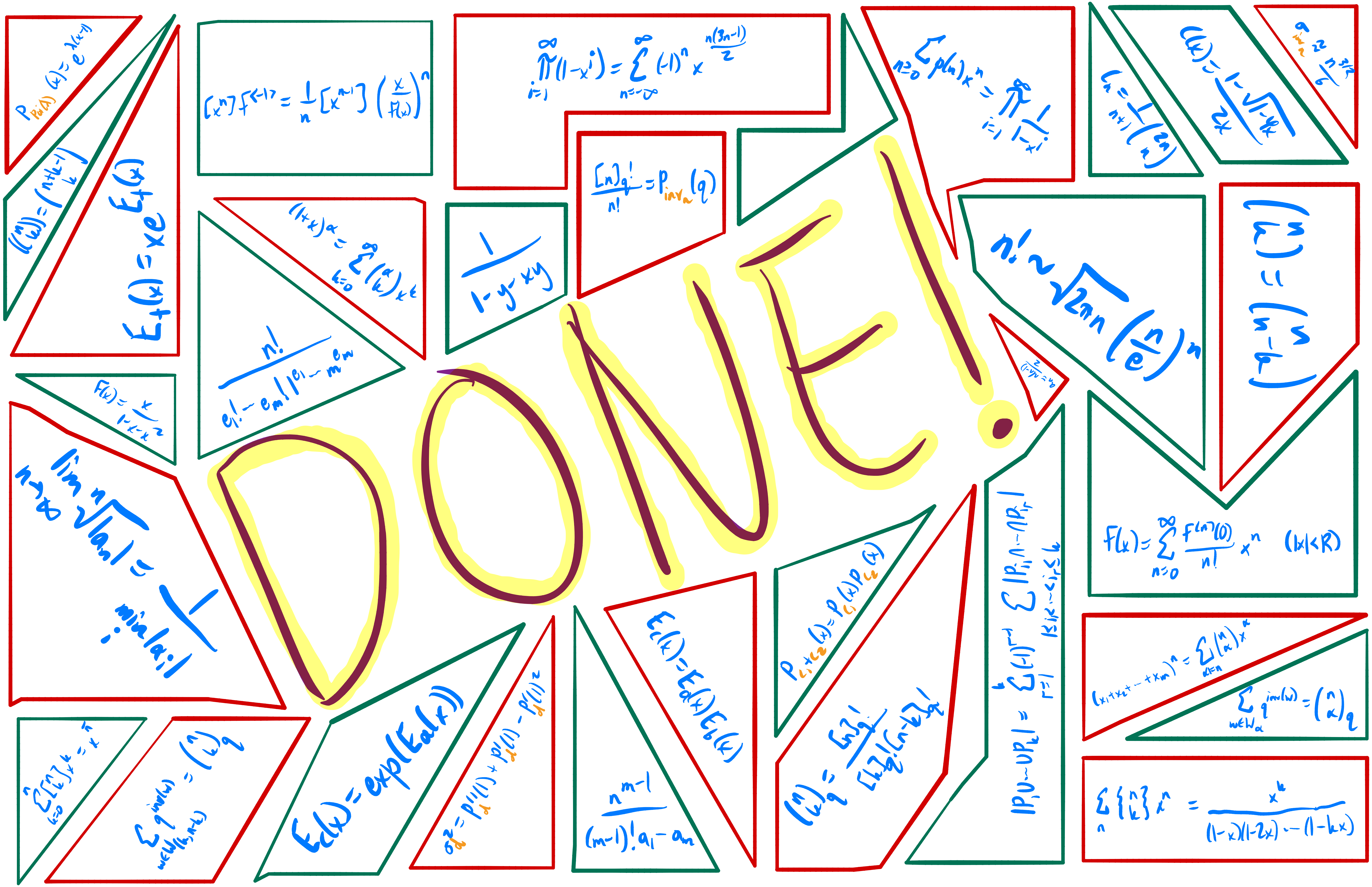
References

- Billey–Swanson '21+
  "Cyclotomic generating functions"
  In preparation.
- Stanley '78
  "Hilbert functions of graded algebras"
  Adv. in Math. **28** (1978)
- Stembridge '94
  "On minuscule representations, plane partitions and involutions in complex Lie groups"
  Duke Math. J. **73** (1994)


DONE!
$P_{Poi(\lambda)}(x) = e^{\lambda(x-1)}$
$[x^n] f^{\langle -1 \rangle} = \frac{1}{n}[x^{n-1}]\left(\frac{x}{f(x)}\right)^n$
$\prod_{i=1}^{\infty}(1-x^i) = \sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{n(3n-1)}{2}}$
$\sum_{n=0} p(n)x^n = \prod_{i=1}^{\infty}\frac{1}{1-x^i}$
$C_n = \frac{1}{n+1}\binom{2n}{n}$
$C(x) = \frac{1-\sqrt{1-4x}}{2x}$
$\left(\binom{n}{k}\right) = \binom{n+k-1}{k}$
$E_T(x) = xe^{E_T(x)}$
$\frac{[n]_q!}{n!} = P_{inv_n}(q)$
$(1+x)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k$
$\frac{1}{1-y-xy}$
$\frac{n!}{e_1!\cdots e_m!1^{e_1}\cdots m^{e_m}}$
$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$
$\binom{n}{k} = \binom{n}{n-k}$
$F(x) = \frac{x}{1-x-x^2}$
$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \frac{1}{\min_i |a_i|}$
$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n \quad (|x|<R)$
$(x_1+x_2+\cdots+x_m)^n = \sum_{\alpha\models n}\binom{n}{\alpha}x^\alpha$
$\sum_{w\in W_\alpha} q^{inv(w)} = \binom{n}{\alpha}_q$
$\sum_n \left\{ {n \atop k} \right\} x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}$
$|P_1\cup\cdots\cup P_k| = \sum_{r=1}^{k}(-1)^{r-1}\sum_{1\le i_1<\cdots<i_r\le k}|P_{i_1}\cap\cdots\cap P_{i_r}|$
$P_{c_1+c_2}(x) = P_{c_1}(x)P_{c_2}(x)$
$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$
$E_c(x) = E_a(x)E_b(x)$
$\frac{n^{m-1}}{(m-1)!a_1\cdots a_m}$
$\sigma_d^2 = P_d''(1) + P_d'(1) - P_d'(1)^2$
$E_c(x) = \exp(E_a(x))$
$\sum_{w\in W(k,n-k)} q^{inv(w)} = \binom{n}{k}_q$
$\sum_{k=0}^{n}\left[{n \atop k}\right]x^k = x^{\overline{n}}$