What Schubert puzzles *really* compute

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Abstract

T. Tao and I, partly with C. Woodward, showed that the bounds on spectra of sums of Hermitian matrices could be described with "puzzles". By comparing with Klyachko's alternate approach, we found the puzzles to compute Schubert structure constants in ordinary and T-equivariant cohomology.

P. Zinn-Justin used H^*_T puzzle pieces to define an "R-matrix", a solution of the Yang-Baxter Equation with spectral parameter. [Maulik-Okounkov '12] construct R-matrices in change-of-stable-basis formulæ on Nakajima quiver varieties, which includes *cotangent bundles to* partial flag manifolds. With this clue PZJ and I showed that for up to 4-step flag manifolds, puzzles compute the structure constants in the product of "motivic Segre classes", more naturally associated to the cotangent bundles. (Our 4-step rule, and 3-step equivariant rule, are not positive.)

These slides are available at `http://math.cornell.edu/~allenk/`

Manifesto: indexing Schubert classes.

The techniques we use come from physics in $1+1$ dimensions, where particles may be *indistinguishable* but are generally at separate *positions*. For this reason we index Schubert classes on $\mathrm{Gr}(k, \mathbb{C}^n)$ not by

$$\underbrace{S_n \big/ (S_k \times S_{n-k})}_{\text{ambiguate positions}} \cong S_n^{S_k \times S_{n-k}} = \{\text{Grassmannian permutations}\}$$

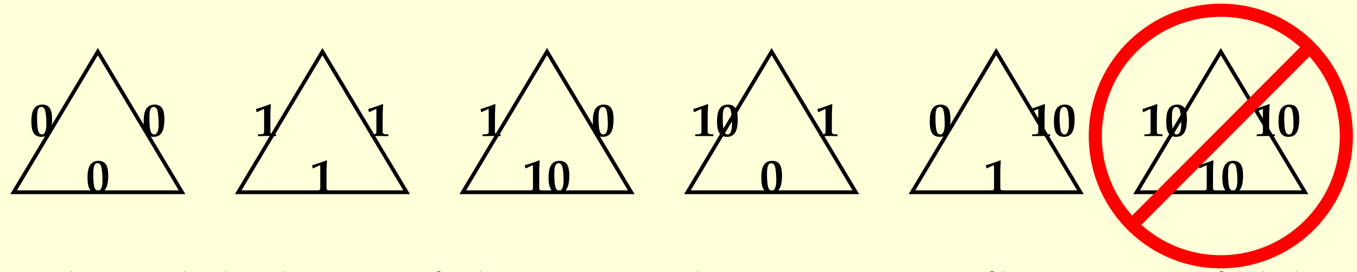
but rather by

$$\underbrace{(S_k \times S_{n-k}) \big\backslash S_n}_{\text{ambiguate values}} \cong {}^{S_k \times S_{n-k}}S_n = \{\text{binary strings with content } 0^k 1^{n-k}\}$$

Similarly, Schubert classes on d-step flag manifolds $Fl(n_1, n_2, \ldots, n_d;\ \mathbb{C}^n)$ will be indexed by strings in $0, \ldots, d$. (The only real confusion arises on full flag manifolds; is a sequence in $1 \ldots n$ a permutation or its inverse string?)

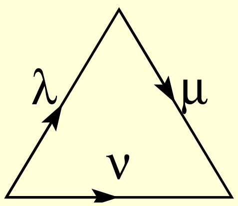
Grassmannian puzzles [KT '03] and their recent developments.

The puzzle pieces are



and in particular involve *three* labels, as if they are about 2-step flag manifolds.

Theorem [Halacheva-K-ZJ '19]. Consider the embedding $\iota : F\ell(j,k;\mathbb{C}^n) \hookrightarrow Gr(j,\mathbb{C}^n) \times Gr(k,\mathbb{C}^n)$. The structure constants $c^{\nu}_{\lambda\mu}$ in the pullback $\iota^*([X_\lambda] \otimes [X_\mu]) = \sum_\nu c^{\nu}_{\lambda\mu}[X_\nu]$, where ν runs over strings in $0, 10, 1$, are the number of puzzles with these boundaries:



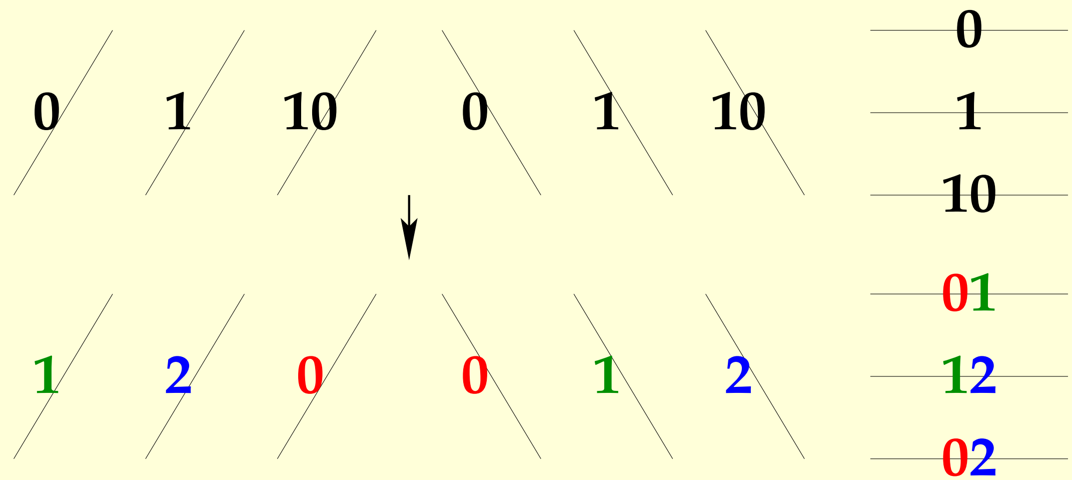
The Green's theorem proof that puzzles with no 10s on boundary (the $j = k$ case) have the same number of 0s on all three sides extends fine if one allows 10-10-10 pieces, in either Δ or ∇ orientation or both.

Theorem. Consider puzzles where we allow in 10-10-10 pieces.

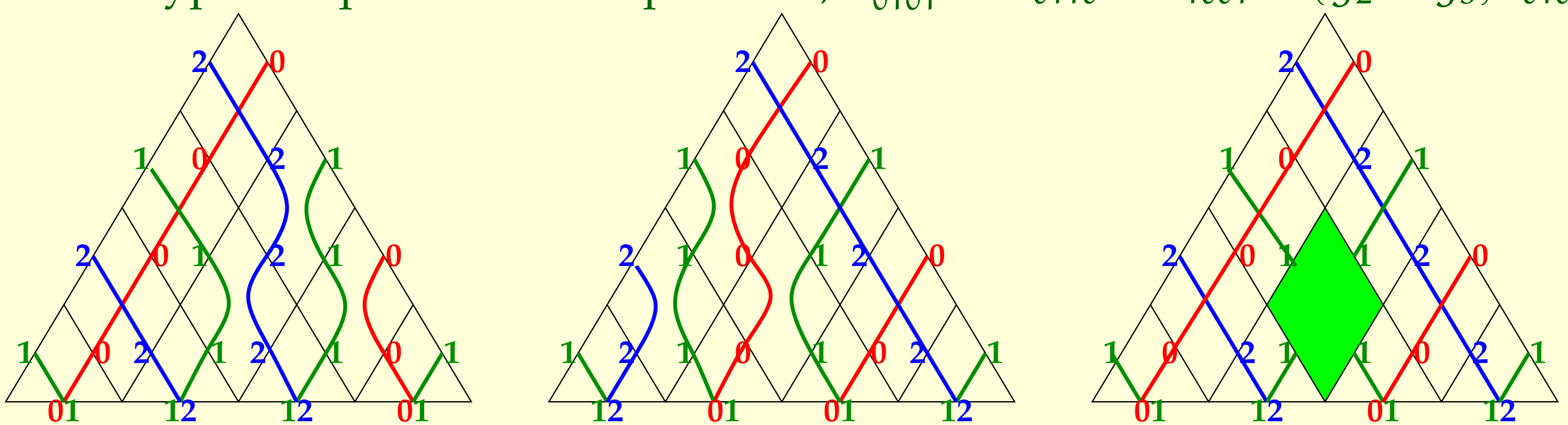
1. [Buch/Tao, appears in Vakil '06.] With Δs, one computes $[\mathcal{O}_{X_\lambda}][\mathcal{O}_{X_\mu}]$.
2. [Wheeler-ZJ '19.] With ∇s, one gets products in the *dual* basis of K-theory.
3. [K-ZJ '21.] Counting both pieces equally computes χ(triple intersections).

Teasing out the conservation law in Grassmannian puzzles.

If we replace the traditional edge labels as pictured, we lose the rotational symmetry (already absent in equivariant calculations) but gain a pipe-dream-like picture.



Here's a typical equivariant computation, $S^2_{0101} = S_{0110} + S_{1001} + (y_2 - y_3)S_{0101}$:



(This is disallowing the 10-10-10s and two new equivariant pieces.)

In this, insertion of one rhombus is an action on length $2n$ strings of $0, 1, 2$. The conservation law matches weight preservation in the map $\mathbb{C}^3 \otimes \mathbb{C}^3 \to \mathrm{Alt}^2\mathbb{C}^3$.

A commuting diagram of correspondences and cohomology.

$$\begin{array}{ccccc}
 & & \overset{\mathrm{graph}(\iota)^T}{\curvearrowright} & & \\
Gr(j,\mathbb{C}^n)\times Gr(k,\mathbb{C}^n) & \xrightarrow[\mathrm{graph}(f)]{} & Fl(k,j+n;\ \mathbb{C}^{2n}) & & Fl(j,k;\ \mathbb{C}^n) \\
\downarrow{\scriptstyle \mathrm{graph}(0)} & & \downarrow{\scriptstyle \mathrm{graph}(0)} & & \downarrow{\scriptstyle \mathrm{graph}(0)} \\
T^*(Gr(j,\mathbb{C}^n)\times Gr(k,\mathbb{C}^n)) & \xrightarrow{C(\mathrm{graph}(f))} & T^*Fl(k,j+n;\ \mathbb{C}^{2n}) & \xrightarrow{//_{Id}Rad(P_n)} & T^*Fl(j,k;\ \mathbb{C}^n)
\end{array}$$

where $f(V^j,V^k) := (V^k\oplus 0, \mathbb{C}^n\oplus V^j)$, and $Rad(P_n) := \left\{\begin{bmatrix} I & 0 \\ \star & I\end{bmatrix}\right\}$.

The *Maulik-Okounkov stable basis* $\{St_w\}$ of $\widetilde{H}^*_{T\times\mathbb{C}^\times}(T^*Fl(n)) \sim H^*_T(Fl(n))[\hbar]$, closely related to the Schubert basis as $\hbar\to\infty$, has the great property that this conormal bundle $C(\mathrm{graph}(f))$ takes $St_\lambda\otimes St_\mu$ to $St_{\lambda\mu}$ where $\mathrm{content}(\lambda) = 1^j 2^{n-j}$, $\mathrm{content}(\mu) = 0^k 1^{n-k}$. As usual it depends on choice of Weyl chamber.

Crossing one Weyl wall is computed by filling in a puzzle rhombus. We cross $\binom{n}{2}$, to get a basis better matched to the Hamiltonian reduction $//_{Id}Rad(P_n)$.

To get to St_λ in the second row, we must start with $SSM_\lambda := St_\lambda / [\text{zero section}]$ in the first, the **Segre-Schwartz-MacPherson class** associated to the Bruhat cell.

Warmup: computing $[X_w]|_v$ using square pipe dreams.

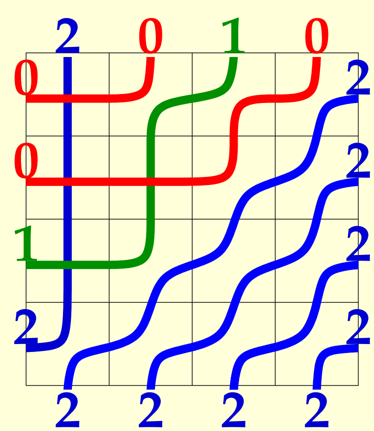
Double Schubert polynomials $\{\mathfrak{S}_w\}$ compute equivariant Schubert classes $\{[X_w]\}$:

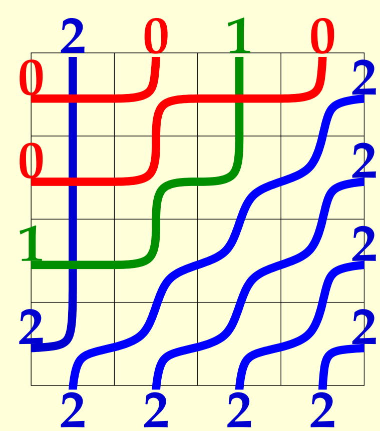
$$[X_w]|_v = \mathfrak{S}_w(y_{v(1)}, \ldots, y_{v(n)}, y_1, \ldots, y_n)$$

We can compute $\mathfrak{S}_\lambda$ itself even for strings (!), as a sum over pipe dreams, using the local (!) rules

$$\begin{matrix} & a & \\ b & + & b \\ & a & \end{matrix} \quad \mapsto \quad \begin{cases} x_{row} - y_{col} & a > b \\ 0 & a \leq b \end{cases} \qquad\qquad \begin{matrix} & a & \\ a & \text{⌟⌜} & b \\ & b & \end{matrix} \quad \mapsto \quad 1$$

Example: $\lambda = 2010$. Put that across **the North side (!)** and its sorted version down the West side, with d everywhere else.





$$(x_1 - y_1)(x_2 - y_1)(x_2 - y_2)(x_3 - y_1) + (x_1 - y_1)(x_1 - y_3)(x_2 - y_1)(x_3 - y_1)$$

Fugacities for defining motivic Segre classes MS_w.

To get an analogue of $\mathfrak{S}_w$ for the K-theory version of SSM classes, **motivic Segre classes**, we still sum over pipe dreams but change the fugacities. Let $z := \exp(x_{row} - y_{col})$, and $K_{\mathbb{C}^\times}(pt) \cong \mathbb{Z}[q^\pm]$ for scaling the cotangent fibers.

$$\begin{matrix} & a & \\ b & + & b \\ & a & \end{matrix} \mapsto \frac{1}{1-q^2z}\begin{cases} q(1-z) & a \neq b \\ 0 & a = b \end{cases} \qquad \begin{matrix} & a & \\ a & \lrcorner\ulcorner & b \\ & b & \end{matrix} \mapsto \frac{1}{1-q^2z}\begin{cases} 1-q^2z & a = b \\ 1-q^2 & a < b \\ (1-q^2)z & a > b \end{cases}$$

Warning #1: in the $q \to 0$ limit of $q^{-\ell(w)}S_w$ to K-theory on the base, this doesn't recover the usual theory of nonreduced pipe dreams – rather than summing over *interior faces* of the pipe dream complex, it corresponds to a sum over *facets* in shelling order, each one contributing only the new part of the simplex.

Warning #2: we've set up signs to limit not to $[\mathcal{O}_{X_w}]$ but $(-1)^{\ell(w)}[\mathcal{O}_{X_w}]$, whose multiplication is positive.

Now, as with $\mathfrak{S}$, we sum over pipe dreams as on the last slide, then specialize $x_i \mapsto y_{v(i)}$ to define the point restrictions $MS_w|_v$ of motivic Segre classes.

Puzzles for multiplying motivic Segre classes, $d \leq 4$.

Theorem [K-ZJ '21]. For $d(\leq 4)$-step flag manifolds M, there is a puzzle formula for multiplication in the $K_{T\times\mathbb{C}^\times}(T^*M)$ basis of motivic Segre classes.

In each case the triangular puzzle pieces correspond to the nonzero matrix entries in the unique-up-to-scale morphism $V_\lambda(z) \otimes V_\mu(q^{h/3}z) \twoheadrightarrow V_\nu(z)$ of modules over a quantized loop algebra $U_q(\mathfrak{x}_{2d}[z^\pm])$, for certain representations λ, μ, ν. (Here h is the dual Coxeter number.)
Meanwhile, the rhomboidal puzzle pieces correspond to the nonzero matrix entries in the R-matrix $V_\lambda(z') \otimes V_\mu(z'') \to V_\mu(z'') \otimes V_\lambda(z')$.

d=1. $\mathfrak{x}_2 = A_2$ and the morphism is $\mathbb{C}^3 \otimes \mathbb{C}^3 \to \mathrm{Alt}^2(\mathbb{C}^3)$.

d=2. $\mathfrak{x}_4 = D_4$ and the morphism is $spin_+ \otimes spin_- \to \mathbb{C}^8$.

d=3. $\mathfrak{x}_6 = E_6$ and the morphism is the trilinear form on $\mathbb{C}^{27}$.

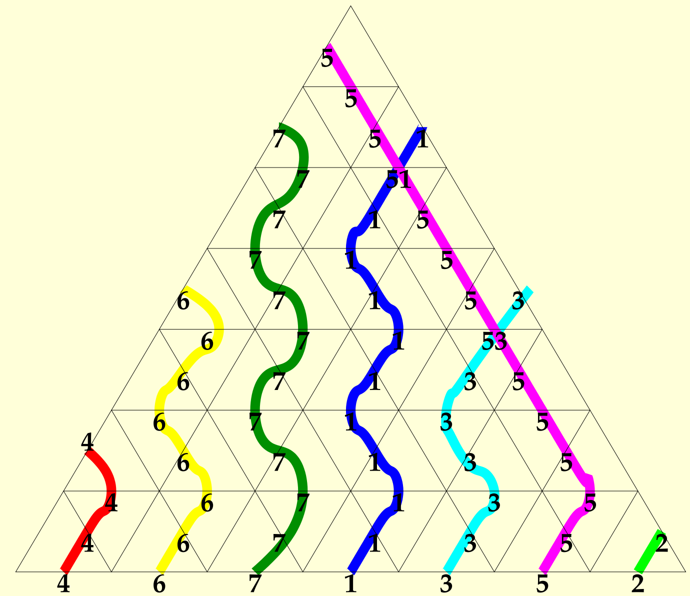
d=4. $\mathfrak{x}_8 = E_8$ and the morphism is the trilinear form on $\mathfrak{e}_8 \oplus \mathbb{C}$.
In this case the representations are not minuscule, one must pick a basis of the zero weight space, and the fugacities can not all be made positive.

Differentiating at $z = 1$ leads to the H^* version. Taking $q \to \infty$ leads to Schubert classes in K-theory. The $d = 3$ rule implies the $d = 1, 2$ but the $d = 4$ doesn't!

Separated descents.

There are two families of minuscule intertwiners, $\bigwedge^a \mathbb{C}^n \otimes \bigwedge^b \mathbb{C}^n \to \bigwedge^{a+b} \mathbb{C}^n$ in type A and $spin_+ \otimes spin_- \to \mathbb{C}^{2n}$ in type D, that give some additional puzzles. It turns out that the $a, b > 1$ cases don't give any more than $a = b = 1$.

Theorem [K-ZJ]. Let $\Delta : Fl(\mathbb{C}^n) \hookrightarrow Fl(\mathbb{C}^n)^2$ be the diagonal, and $\pi_{\leq k} : Fl(\mathbb{C}^n) \twoheadrightarrow Fl(1, \ldots, k; \mathbb{C}^n)$, $\pi_{\geq k} : Fl(\mathbb{C}^n) \twoheadrightarrow Fl(k, \ldots, n; \mathbb{C}^n)$ be the projections. Then $\Delta^*(\pi_{\leq k} \times \pi_{\geq k})^*(MS_\lambda \otimes MS_\mu)$ can be computed using puzzles like these on the right:



Warning. Unlike the Schubert or $[\mathcal{O}_{X_w}]$ pullbacks, we have $\pi^*_{\leq k}(S_\lambda) = \sum_{w \in W_P \lambda} (-1)^{\ell(w)-\ell(\lambda)} S_w$, more like the $[\mathcal{I}_{X_w}]$ pullback, with a similar statement for $\pi^*_{\geq k}$. So the puzzle formula is *not* computing special cases of $\Delta^*(MS_w \otimes MS_v)$.

D. Huang has given a tableaux-based formula, for ordinary Schuberts in H^*.

The type D puzzles are work in progress, but give only a little more, $Fl(n_1, \ldots, n_m; \mathbb{C}^n) \hookrightarrow Fl(n_1, \ldots, \underline{n_{k-1}, n_k}; \mathbb{C}^n) \times Fl(\underline{n_{k-1}, n_k}, \ldots, n_m; \mathbb{C}^n)$ where the sadly minimal overlap is underlined.