Positroids, knots, and q, t-Catalan numbers

Pavel Galashin (UCLA)

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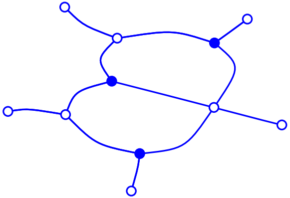
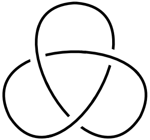
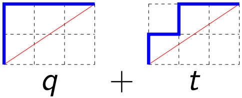
Joint work with Thomas Lam (arXiv:2012.09745)







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- Reason: Schubert decomposition.

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$$\mathcal{P}(Z; q, t) := \sum_{i,p\in\mathbb{Z}} q^{p-\frac{i}{2}} t^{\frac{d-i}{2}} \dim H^{i,(p,p)}(Z), \quad \text{where } d := \dim Z.$$

Positroid varieties

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- Point count? Poincaré polynomial? $\mathcal{P}(\Pi^\circ_{k,n}; q, t)$?

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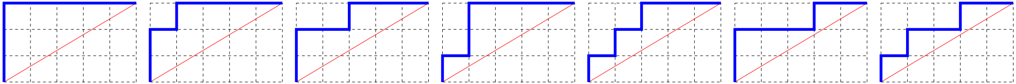

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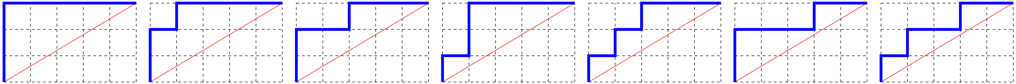

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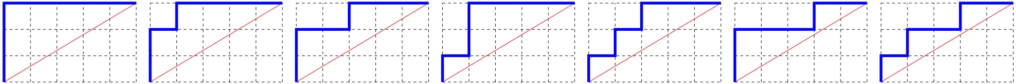

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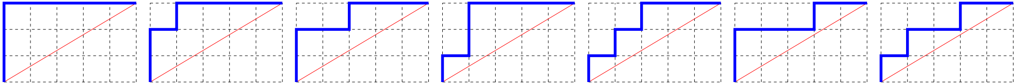

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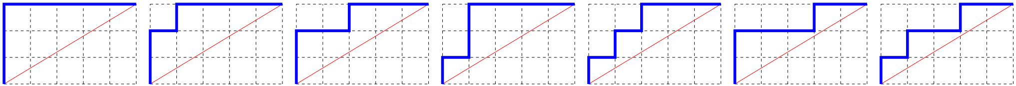

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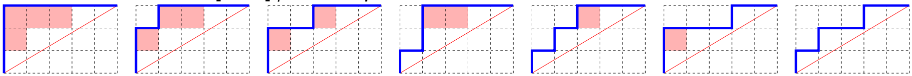

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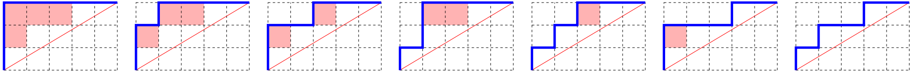

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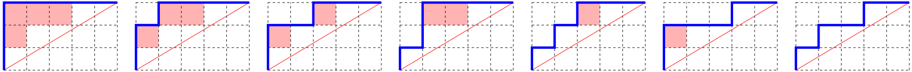

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Theorem (G.–Lam (2020))

Let $\gcd(k,n)=1$. *Then the point count and the Poincaré polynomial of* $\Pi^\circ_{k,n}$ *are*

$$\#\Pi^\circ_{k,n}(\mathbb{F}_q) = (q-1)^{n-1}\cdot C'_{k,n-k}(q), \qquad \mathcal{P}(\Pi^\circ_{k,n};q) = (q+1)^{n-1}\cdot C''_{k,n-k}(q^2).$$

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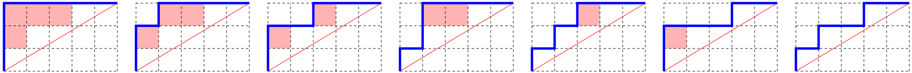

$$\sum_{P \in \mathrm{Dyck}_{a,b}} q^{\mathrm{area}(P)} = q^4 + q^3 + q^2 + q^2 + q^1 + q^1 + q^0$$

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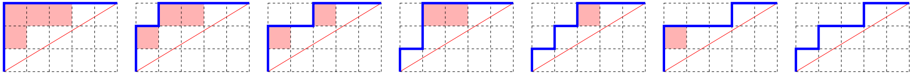

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Rational q, t-Catalan numbers: (introduced by Garsia–Haiman (1996) and Loehr–Warrington (2009))

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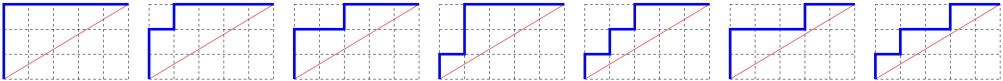

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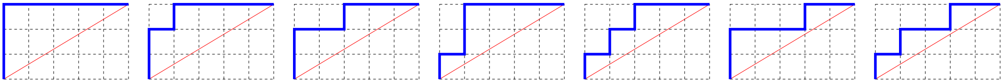

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[LS16] Thomas Lam and David E. Speyer. Cohomology of cluster varieties. I. Locally acyclic case. arXiv:1604.06843.

[Sco06] J. S. Scott. Grassmannians and cluster algebras. *Proc. Lond. Math. Soc. (3)*, 92(2):345–380, 2006.

[GL19] Pavel Galashin and Thomas Lam. Positroid varieties and cluster algebras. arXiv:1906.03501.

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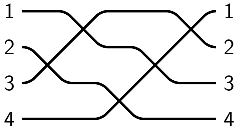

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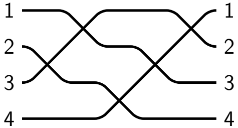

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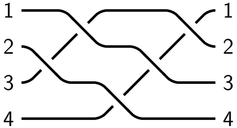

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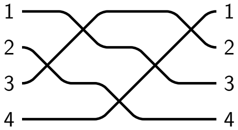

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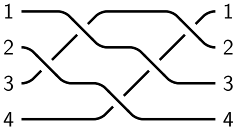

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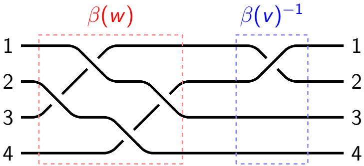

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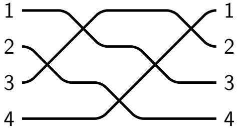

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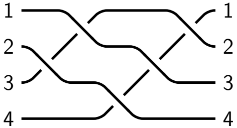

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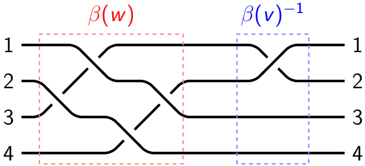

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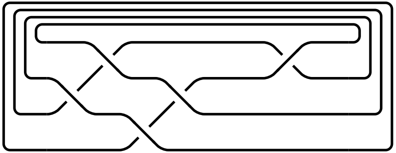

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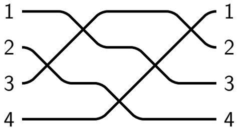

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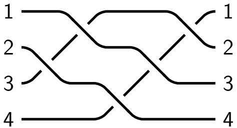

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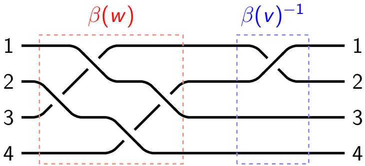

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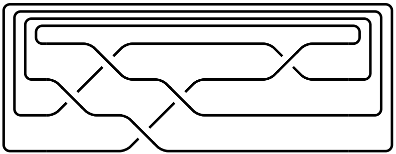
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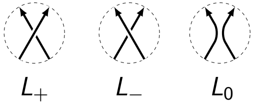

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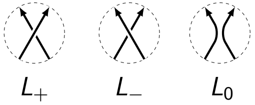

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Let $c(wv^{-1}) = 1$. $\#(R^{\circ}_{v,w}/T)(\mathbb{F}_q) =$ *top a-degree term of* $P(\hat{\beta}_{v,w}; a, q)$;

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For $c(wv^{-1}) \geqslant 1$, take the T-equivariant cohomology of $R^{\circ}_{v,w}$ with compact support instead.

Given a link L, the HOMFLY polynomial $P(L; a, q)$ is defined by $P(\bigcirc) = 1$ and

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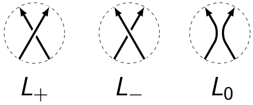

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Thanks!