Coxeter-like elements, Schubert geometry, and multiplicity-freeness in algebraic combinatorics

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Based on joint work with:

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General goal: Understand global and local properties of Schubert varieties.

- Singularities and finer measures such as Hilbert-Samuel multiplicity $\Rightarrow$ *commutative algebra*
- Kazhdan-Lusztig polynomials $\Rightarrow$ *Hecke algebras*

We are interested in combinatorial descriptions of the number/classification.

Schubert varieties II: overview (continued)

The problem set about Schuberts that we study has **no finite algorithm**, *a priori*.

Let $X = \mathrm{Flags}(\mathbb{C}^n)$. GL_n and upper triangulars $\mathrm{B} \subset \mathrm{GL}_n$ act on X.

Definition: *Schubert varieties* = B-orbit closures X_w, $w \in S_n$.

- $\dim(X_w) = \ell(w) = \{1 \leq i < j \leq n : w(i) > w(j)\}$
- X_w is a union of B-orbits and hence has a B-action.
- X_w has a "secret" group action of a "Levi" depending on w.

Schubert varieties III: the main problem

Left descents: $J(w) = \{1 \leq i \leq n : i+1 \text{ appears left of } i \text{ in } w\}$

Example: $w = 31524 \Rightarrow J(w) = \{2, 4\}$.

Let $I \subseteq J(w)$; this gives the *subdivision*

$$D := [n] - I = \{d_1 < d_2 < \ldots < d_k\}.$$

Declare $d_0 := 0, d_{k+1} = n$.

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Definition: The *Levi-subgroup* of GL_n for I is the block submatrix

$$L_I \cong \mathrm{GL}_{d_1-d_0} \times \mathrm{GL}_{d_2-d_1} \times \cdots \times \mathrm{GL}_{d_{k+1}-d_k}.$$

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Fact: L_I acts on X_w.

Main definition: If $I \subseteq J(w)$, X_w is *I-spherical* if X_w has a dense orbit of a Borel in L_I.

Main problem: Which X_w are I-spherical?

Probabilistic combinatorics

Definition: $w \in S_n$ is *proper* if $\ell(w) \leq n + \binom{\#J(w)+1}{2}$.

Actually, for $1 \leq n \leq 10$, proper permutations are not rare:

$$1, 2, 6, 24, 120, 684, 4348, 30549, 236394, 2006492, \ldots$$

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Theorem: (Brewster-Hodges-Y. '20)

$$\Pr[w \text{ is proper}] \to 0 \text{ as } n \to \infty.$$

Thus

$$\lim_{n\to\infty} \Pr[w \in S_n, X_w \text{ is } L_I\text{-spherical for some } I \subset J(w)] \to 0.$$

Proof idea: w not proper means $\dim(X_w)$ is too large for a Borel orbit to be dense. Now use the second moment method. □

Split-symmetry in algebraic combinatorics

A paradigm in algebraic combinatorics:

$$\text{Symmetric polynomials} \hookrightarrow \text{Polynomials}$$
$$\text{Schur} \qquad \text{Schubert, Grothendieck,....}$$

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Definition: (Hodges-Yong, '20) Π_D = ring of polynomials symmetric in $X_i := \{x_{d_{i-1}+1}, \ldots, x_{d_i}\}$ for $1 \leq i \leq k+1$.

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Let Par_n = partitions with at most n parts. Basis of Π_D is:

$$s_{\lambda^1,\ldots,\lambda^k} := s_{\lambda^1}(X_1)\cdots s_{\lambda^k}(X_k)$$

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Ex. (Vieta's formulas, a reinterpretation)

$$f = s_{(n-1)}(x_1)s_{\emptyset}(x_2,\ldots,x_n) + s_{(n-2)}(x_1)s_{(1)}(x_2,\ldots,x_n) + \cdots + s_{\emptyset}(x_1)s_{(1^{n-1})}(x_2,\ldots,x_n).$$

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is *D-multiplicity-free* if $c_{\lambda^1,\ldots,\lambda^k} \in \{0,1\}$ for all $(\lambda^1,\ldots,\lambda^k) \in \mathrm{Par}_D$.

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- $D = \emptyset$: mult.-freeness of symmetric functions into Schurs (e.g., Stembridge '01, ..., S. Gao-Hodges-Orelowitz '20).
- $D = [n-1]$: mult.-freeness of monomial expansion (e.g., Fink-Mészáros-St. Dizier '19 for Schubert, Hodges-Y. '20 for key polynomials)

Key polynomials (a.k.a. Demazure characters)

The *Demazure operator* is

$$\begin{aligned}\pi_j : \mathrm{Pol}_n &\to \mathrm{Pol}_n \\ f &\mapsto \frac{x_j f - x_{j+1} s_j f}{x_j - x_{j+1}},\end{aligned}$$

where $s_j f := f(x_1, \ldots, x_{j+1}, x_j, \ldots, x_n)$.

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A *weak composition* is $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$.

Def: (Lascoux-Schützenberger, '89) The *key polynomial* κ_α is

$$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \text{if } \alpha \text{ is weakly decreasing.}$$

Otherwise,

$$\kappa_\alpha = \pi_j(\kappa_{\widehat{\alpha}}) \text{ where } \widehat{\alpha} = (\alpha_1, \ldots, \alpha_{j+1}, \alpha_j, \ldots, \alpha_n) \text{ and } \alpha_{j+1} > \alpha_j.$$

Example: $\alpha = (0, 2, 0, 1)$

$$\kappa_\alpha = x_2^2x_4+x_2^2x_3+x_1x_2x_4+x_1^2x_4+x_1x_3x_2+x_1^2x_2+x_1^2x_3+x_1x_3x_2+x_1x_2^2$$

is symmetric in $X_1 = \{x_1, x_2\}, X_2 = \{x_3, x_4\}$, and

$$\kappa_\alpha = s_{(2)}(x_1, x_2)s_{(1)}(x_3, x_4) + s_{(2,1)}(x_1, x_2)s_{\emptyset}(x_3, x_4)$$

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Key polynomials II

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Theorem: (Hodges-Y. '20) Let $w \in S_n$, $I \subseteq J(w)$. Then X_w is L_I-spherical $\iff$ $\kappa_{w\lambda}$ is D-mult.-free for all $\lambda \in \text{Par}_n$.

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Conjecture: (Hodges-Y. '20) Sphericality holds if $\kappa_{w\rho_n}$ is D-multiplicity-free, where $\rho = (n, n-1, \ldots, 3, 2, 1)$.

Coxeter(-like) elements and reduced words

Let W_P be a parabolic subgroup of $W(= S_n)$.

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Let $\mathrm{Red}(w)$ be the set of reduced words for w.

In [Hodges-Yong, '20], a different def. of I-spherical was given.

Definition: Let $w \in S_n$ and $I \subset J(w)$. w is *I-spherical* if $R = s_{i_1}s_{i_2}\cdots s_{i_{\ell(w)}} \in \mathrm{Red}(w)$ exists such that

(S.1') s_{d_i} appears at most once in R

(S.2') $\#\{m : d_{t-1} < i_m < d_t\} < \binom{d_t-d_{t-1}+1}{2}$ for $1 \leq t \leq k+1$.

Theorem: (Gao-Hodges-Y., '21+) The two definitions of I-spherical agree for $W = S_n$.

We are working towards:

Conjecture: (Hodges-Y., '20, Gao-Hodges-Y., '21)
Let $w \in S_n$ and $I \subset J(w)$. X_w is L_I-spherical if and only if w is I-spherical.

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These ideas have some extensions to other finite types; see [Hodges-Y., '20] for discussion, including evidence using work of

- Avdeedv-Petukhov '14
- Can-Hodges '18
- Hodges-Lakshmibai '18
- Karuppuchamy '13
- Magyar-Weyman-Zelevinsky '99, ...

Some data; pattern avoidance

Data: All $w \in S_n$ are $J(w)$-spherical, if $n \leq 4$. In S_5 the non-examples are

$$24531, 25314, 25341, 34512, 34521, 35412, 35421, 42531,$$

$$45123, 45213, 45231, 45312, 52314, 52341, 53124, 53142,$$

$$53412, 53421, 54123, 54213, 54231.$$

Example: $w = 24531 \Rightarrow J(w) = \{1, 3\}$ and $w_0(J(w))w = s_1 \cdot s_3 \cdot 24531 = 13542 = \underline{s_2} s_4 s_3 \underline{s_2}$ (not Coxeter);

Definition: $v \in S_N$ *avoids* $u \in \mathfrak{S}_n$ if there does not exist $\phi_1 < \phi_2 < \ldots < \phi_n$ such that $v(\phi_1), \ldots, v(\phi_n)$ are in the same relative order as $u(1), \ldots, u(n)$.

Conjecture: (Hodges-Y., '20) X_w is $J(w)$-spherical if and only if w pattern avoids all of the permutations listed above.

Conclusions and summary

In this talk, we discussed when a Schubert variety is spherical.

Spherical variety theory generalizes that of toric varieties (see papers of, e.g., Brion-Luna-Vust '86, Luna '01, Perrin '14).

We offer such (conjectural) classifications. Our work makes introduces problems/relations to:

- Probabilistic combinatorics
- Combinatorics of polynomials
- Coxeter combinatorics
- Pattern avoidance

Thank you!