A combinatorial Chevalley formula for semi-infinite flag manifolds and its applications

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Joint work with Satoshi Naito (Tokyo Institute of Technology) and Daisuke Sagaki (Tsukuba University).

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Notation

G semisimple Lie group over $\mathbb{C}$.

$T \subset B \subset G$, T maximal torus, B Borel subgroup.

N unipotent radical, $B = TN$.

P weight lattice, ω_i fundamental weights ($i \in I$),
P^+ dominant weights.

Q root lattice, $Q^\vee$ coroot lattice, α_i simple roots ($i \in I$).

$\mathbb{Z}[P] = R(T) = \bigoplus_{\lambda \in P} \mathbb{Z}\mathbf{e}^\lambda$.

W finite Weyl group, s_i simple reflections, $w_\circ$ longest element.

Main goals

Chevalley formula for $K_T(G/B)$ (as module over $K_T(\mathrm{pt}) = \mathbb{Z}[P]$), for any $w \in W$ and $\lambda \in P$, where $\mathcal{L}_\lambda := G \times_B \mathbb{C}_{-\lambda}$:

$$[\mathcal{L}_\lambda] \cdot [\mathcal{O}_{X_w}] = \sum_{v \in W,\, \mu \in P} c_{w,v}^{\lambda,\mu}\, \mathbf{e}^{\mu}\, [\mathcal{O}_{X_v}]\,, \qquad c_{w,v}^{\lambda,\mu} \in \mathbb{Z}\,.$$

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- Chevalley formulas for $QK_T(G/B)$ and $QK_T(G/P)$;
- applications: more explicit computations and results in type A.

The semi-infinite flag manifold

$\mathbf{Q}_G^{\mathrm{rat}}$ is the reduced ind-scheme associated to

$$G(\mathbb{C}(\!(z)\!))/(T \cdot N(\mathbb{C}(\!(z)\!)))\,.$$

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$K_{T\times\mathbb{C}^*}(\mathbf{Q}_G)$ has a $\mathbb{Z}[q,q^{-1}][P]$-basis of classes $[\mathcal{O}_{\mathbf{Q}_G(x)}]$ of the structure sheaves of $\mathbf{Q}_G(x)$, for $x \in W_{\mathrm{aff}}^{\geq 0} = W \ltimes Q^{\vee,+}$.

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[L.-Naito-Sagaki]:

- translate the Chevalley formulas for $\lambda \in P^+$ and $\lambda \in P^-$ from quantum LS paths to the quantum alcove model (below);
- generalize the new formulas to arbitrary $\lambda \in P$, via combinatorics of the quantum alcove model.

Quantum Bruhat graph on the finite Weyl group

The quantum Bruhat graph on W, denoted $\mathrm{QBG}(W)$, is the directed graph with labeled edges

$$w \xrightarrow{\alpha} ws_\alpha\,, \qquad \text{where}$$

$$\begin{aligned} &\ell(ws_\alpha) = \ell(w) + 1 \quad \text{(covers of Bruhat order)}\,, \qquad \text{or} \\ &\ell(ws_\alpha) = \ell(w) - 2\mathrm{ht}(\alpha^\vee) + 1\,. \end{aligned}$$

(If $\alpha^\vee = \sum_i c_i \alpha_i^\vee$, then $\mathrm{ht}(\alpha^\vee) := \sum_i c_i$.)

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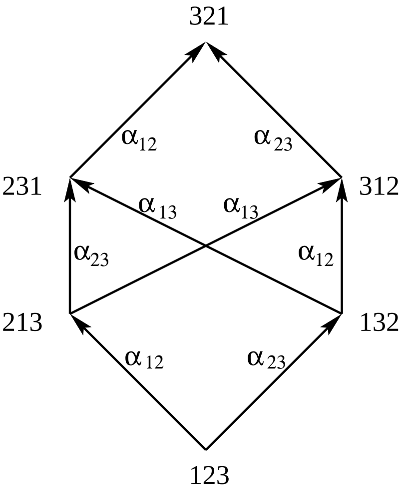
It has a natural lift to the (covers of the) Bruhat order on the affine Weyl group W_{aff} [Lam-Shimozono].

Hasse diagram of the Bruhat order for S_3:



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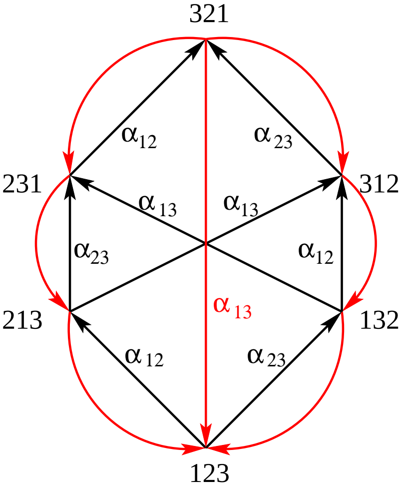

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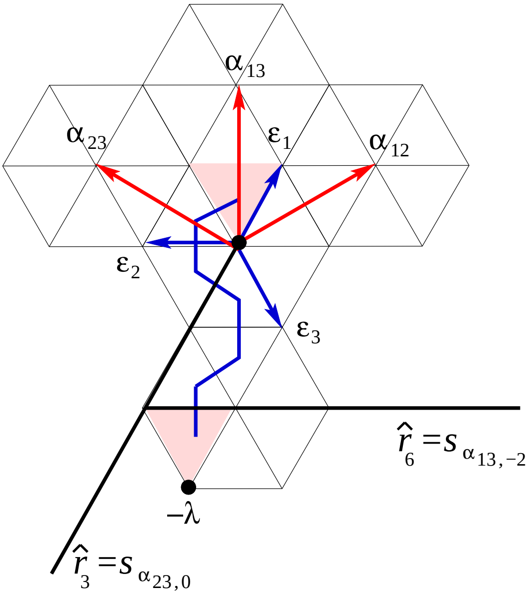
The latter gives a shortest sequence of adjacent alcoves from $A_\circ$ to $A_\circ - \lambda$.

Example. Type A_2, $\lambda = (3, 1, 0) = 3\varepsilon_1 + \varepsilon_2$,
$\Gamma = (\ (1, 2),\ (1, 3),\ (2, 3),\ (1, 3),\ (1, 2),\ (1, 3)\ )$.



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The main structure structure: *w*-admissible subsets

$$\mathcal{A}(w, \Gamma) := \{A \,:\, \pi(w, A) \text{ path in } \mathrm{QBG}(W)\}\,.$$

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- $\mathrm{height}(w, A) := \sum_{j \in A^-} \mathrm{sgn}(\beta_j)\widetilde{l}_j$, for $\widetilde{l}_i := \langle \lambda, \beta_i^\vee \rangle - l_i$.

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$$\cdot \left[\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(w,A) t_{\xi + \mathrm{down}(w,A) + \iota(\boldsymbol{\chi})})}\right],$$

where $n(A)$, for $A = \{j_1 < \cdots < j_s\}$, is the number of negative roots in $\{\beta_{j_1}, \ldots, \beta_{j_s}\}$.

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$QK_T(G/B)$ (small) is defined on $K_T(G/B) \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q][P]$ [Anderson-Chen-Tseng].

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Consider variables Q_i for $i \in I$, and let

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$QK_T(G/B)$ (small) is defined on $K_T(G/B) \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q][P]$ [Anderson-Chen-Tseng].

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Given $\xi = d_1\alpha_1^\vee + \cdots + d_r\alpha_r^\vee$ in $Q^{\vee,+}$, let $Q^\xi := Q_1^{d_1} \cdots Q_r^{d_r}$.

The Chevalley formula in $QK_T(G/B)$

Theorem. [L.-Naito-Sagaki, conjecture by L.-Postnikov] Let $k \in I$, and fix a $(-\omega_k)$-chain of roots $\Gamma(-\omega_k)$. Then, in $QK_T(G/B)$, we have the cancellation-free formula:

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$$[\mathcal{O}^{s_k}] \cdot [\mathcal{O}^{w}] = (1 - \mathbf{e}^{w(\omega_k)-\omega_k})\,[\mathcal{O}^{w}] +$$

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Proof: Translate the (anti-dominant) Chevalley formula for the semi-infinite flag manifold via Kato's isomorphism; cf. Peterson's isomorphism and its extension to K-theory [Peterson, Lam-Shimozono, Lam-Li-Mihalcea-Shimozono, Ikeda-Iwao-Maeno].

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Theorem. [Kato] There is a $\mathbb{Z}[P]$-module isomorphism respecting products

$$QK_T(G/B) \xrightarrow{\simeq} K'_T(\mathbf{Q}_G) \subset K_T(\mathbf{Q}_G)\,.$$

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- perform all cancellations via a sign-reversing involution.

Type A_{n-1}: $QK(Fl_n)$

Theorem. [L.-Naito-Sagaki, conjecture by L.-Maeno] The quantum Grothendieck polynomials [L.-Maeno] represent Schubert classes in $QK(Fl_n)$.

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Theorem. [L.-Naito-Sagaki] For every k, v and parabolic coset $\sigma W_{I\setminus\{k\}}$ not containing v, there exist unique d and $w \in \sigma W_{I\setminus\{k\}}$ (constructed explicitly), such that $N_{s_k,w}^{v,d} = \pm 1$ (sign determined).

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Theorem. [L.-Naito-Sagaki] In the expansion of $[\mathcal{O}^{s_k}] \cdot [\mathcal{O}^w]$ there is a minimum and a maximum degree (with respect to the componentwise order), which are constructed explicitly.