Ideals of skew-symmetric matrix Schubert varieties

Brendan Pawlowski (University of Southern California)
joint with Eric Marberg

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Matrix Schubert varieties

- $[n] = \{1, 2, \ldots, n\}$ and M_n the space of $n \times n$ matrices over an algebraically closed field $\mathbb{K}$.
- A_{IJ} = submatrix of A in rows I and columns J.

The matrix Schubert variety of $w \in S_n$ (identified with its permutation matrix) is

$$X_w = \{A \in M_n : \operatorname{rank} A_{[i][j]} \leq \operatorname{rank} w_{[i][j]} \forall i, j \in [n]\}.$$

Example

If $w = 132 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ then $(\operatorname{rank} w_{[i][j]})_{i,j \in [3]} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \boxed{1} & 2 \\ 1 & 2 & 3 \end{pmatrix}$.

Boxed rank condition implies all others, so

$$X_{132} = \{A \in M_3 : \operatorname{rank} A_{[2][2]} \leq 1\} = \{(a_{ij})_{i,j \in [3]} : a_{11}a_{22} - a_{12}a_{21} = 0\}.$$

- The X_w are closures of two-sided Borel orbits on M_n.
- Torus-equivariant cohomology/K-theory classes of X_w are Schubert/Grothendieck polynomials.

Matrix Schubert varieties

- Let $\mathbb{K}[M_n] = \mathbb{K}[v_{ij} : i, j \in [n]]$ and $V = (v_{ij})_{i,j \in [n]}$.
- Fulton: the prime ideal $I(X_w) \subseteq \mathbb{K}[M_n]$ is generated by the minors $\det V_{IJ}$ where $I \subseteq [i], J \subseteq [j]$ and $|I| = |J| = \operatorname{rank} w_{[i][j]} + 1$, for all $i, j \in [n]$.
- Ex.: $I(X_{132}) = (\det V_{[2][2]}) = (v_{11}v_{22} - v_{12}v_{21})$.

Knutson and Miller:

- These minors form a Gröbner basis of $I(X_w)$ (with respect to, say, reverse lex order on $\mathbb{K}[v_{ij}]$).
- Recall that the <u>initial ideal</u> $\operatorname{init} I(X_w)$ is the monomial ideal generated by the leading terms of each $f \in I(X_w)$.
- $\operatorname{init} I(X_w)$ is generated by squarefree monomials, and its Stanley-Reisner complex is shellable with facets in bijection with the <u>pipe dreams</u> of w.

Skew-symmetric matrix Schubert varieties

- $\mathcal{I}_n^{\mathrm{FPF}}$ = fixed-point-free involutions in S_n.
- e.g. $\mathcal{I}_4^{\mathrm{FPF}} = \{(1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$.
- SSM_n = skew-symmetric matrices over $\mathbb{K}$.

Given $z \in \mathcal{I}_n^{\mathrm{FPF}}$ define the <u>skew-symmetric matrix Schubert variety</u>

$$\mathrm{SSX}_z = \mathrm{SSM}_n \cap X_z = \{A \in \mathrm{SSM}_n : \operatorname{rank} A_{[i][j]} \leq \operatorname{rank} z_{[i][j]} \forall i, j\}.$$

Example

If $z = (1,4)(2,3) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ then $(\operatorname{rank} z_{[i][j]}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ \boxed{0} & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ and

$$\mathrm{SSX}_{(1,4)(2,3)} = \{A \in \mathrm{SSM}_4 : \operatorname{rank} A_{[3][1]} \leq 0\} = \left\{ \begin{pmatrix} 0 & 0 & 0 & -a \\ 0 & 0 & -b & -c \\ 0 & b & 0 & -d \\ a & c & d & 0 \end{pmatrix} \right\}.$$

Skew-symmetric matrix Schubert varieties

Why study SSX_z?

- SSX_z = Borel orbit closure on SSM_n (hence irreducible)
- Connection to spherical orbits:

  $$\mathrm{SSM}_n \cap \mathrm{GL}_n \simeq \{\text{symplectic bilinear forms on } \mathbb{K}^n\} \simeq \mathrm{GL}_n / \mathrm{Sp}_n,$$

  and B-orbits on $\mathrm{GL}_n / \mathrm{Sp}_n \longleftrightarrow \mathrm{Sp}_n$-orbits on GL_n / B.
- Ideals $I(\mathrm{SSX}_z)$ generalize families of Pfaffian ideals studied by De Negri, De Negri–Sbarra, Herzog–Trung, Jonsson–Welker, Raghavan–Upadhyay, ...

What is the prime ideal $I(\mathrm{SSX}_z)$? Notation:

- Let $\mathcal{U} = \begin{pmatrix} 0 & -u_{21} & -u_{31} & \cdots \\ u_{21} & 0 & -u_{32} & \cdots \\ u_{31} & u_{32} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$.
- $I(\mathrm{SSX}_z) \subseteq \mathbb{K}[\mathrm{SSM}_n] = \mathbb{K}[u_{ij} : 1 \leq j < i \leq n]$.

Ideals of skew-symmetric matrix Schubert varieties

Problem: minors of skew-symm. $\mathcal{U}$ need not generate radical ideals.

Example

$$\{A \in \mathrm{SSM}_2 : \operatorname{rank} A \leq 1\} = \left\{A = \left(\begin{smallmatrix} 0 & -a \\ a & 0 \end{smallmatrix}\right) \in \mathrm{SSM}_2 : \det A = a^2 = 0\right\}.$$

Solution: if $A \in \mathrm{SSM}_n$ then $\det(A) = \mathrm{pf}(A)^2$, so use Pfaffians.

Theorem (Marberg–Pawlowski)

Let $z \in \mathcal{I}_n^{\mathrm{FPF}}$. The prime ideal $I(\mathrm{SSX}_z)$ is generated by all Pfaffians $\mathrm{pf}(\mathcal{U}_{SS})$ where $S \subseteq [i]$ and $|S \cap [j]| > \operatorname{rank} z_{[i][j]}$ for all $j < i \leq n$.

- Linear algebra exercise to show that SSX_z is the zero locus of these Pfaffians.
- Harder to show they generate a prime ideal.

Gröbner bases

- Equip $\mathbb{K}[\mathrm{SSM}_n] = \mathbb{K}[u_{ij} : 1 \le j < i \le n]$ with reverse lex order.
- Recall that the initial ideal $\mathrm{init}\, I(\mathrm{SSX}_w)$ is the monomial ideal generated by the leading terms of each $f \in I(\mathrm{SSX}_w)$...
- ... and a generating set of $I(\mathrm{SSX}_z)$ is a Gröbner basis if its leading terms generate the initial ideal $\mathrm{init}(I(\mathrm{SSX}_z))$.

Example

Consider $\mathrm{SSX}_z = \{A \in \mathrm{SSM}_6 : \mathrm{rank}\, A_{[5][3]} \le 2\}$. Ideal is generated by $\mathrm{pf}(\mathcal{U}_{SS})$ where $S \subseteq [5]$ and $|S \cap \{1,2,3\}| > 2$:

$$f_1 = \mathrm{pf}(\mathcal{U}_{1234,1234}) = u_{32}u_{41} - u_{31}u_{42} + u_{21}u_{43}$$
$$f_2 = \mathrm{pf}(\mathcal{U}_{1235,1235}) = u_{32}u_{51} - u_{31}u_{52} + u_{21}u_{53}$$

- Problem: $u_{41}f_2 - u_{51}f_1 = u_{31}u_{42}u_{51} + \cdots$, but $u_{31}u_{42}u_{51} \notin (u_{32}u_{41}, u_{32}u_{51})$, so $\{f_1, f_2\}$ is not a Gröbner basis.
- Worse: $\mathrm{init}\, I(X) = (u_{32}u_{41}, u_{32}u_{51}, \underline{u_{31}u_{42}u_{51}})$, so no set of Pfaffians $\mathrm{pf}(\mathcal{U}_{SS})$ can be a Gröbner basis for $I(X)$!

Initial ideal of $I(\mathrm{SSX}_z)$

- $I(\mathrm{SSX}_z)$ is generated by Pfaffians, but also contains various minors $\det \mathcal{U}_{AB}$.
- If $A = \{a_0 < \cdots < a_r\}$ and $B = \{b_0 > \cdots > b_r\}$, lead term of $\pm \det \mathcal{U}_{AB}$ is antidiagonal $u_{a_0 b_0} \cdots u_{a_r b_r}$ with transformations $u_{ji} \mapsto u_{ij}$ if $j < i$ and $u_{ii} \mapsto 0$ applied.
- Define u_{AB} as the squarefree radical of this monomial.

Example

- $A = \{1, 2, 3\}, B = \{3, 2, 1\} \rightsquigarrow u_{13} u_{22} u_{31} \rightsquigarrow u_{AB} = 0$
- $A = \{1, 3, 4\}, B = \{4, 2, 1\} \rightsquigarrow u_{14} u_{32} u_{41} \rightsquigarrow u_{41}^2 u_{32} \rightsquigarrow u_{AB} = u_{41} u_{32}$
- $A = \{1, 4, 5\}, B = \{3, 2, 1\} \rightsquigarrow u_{13} u_{42} u_{51} \rightsquigarrow u_{AB} = u_{31} u_{42} u_{51}$

Theorem (Marberg–Pawlowski)

$\mathrm{init}(I(\mathrm{SSX}_z))$ *is generated by all monomials* u_{AB} *for* $A \subseteq [i], B \subseteq [j]$ *with* $|A| = |B| = \mathrm{rank}\, z_{[i][j]} + 1$, *for all* $1 \leq j < i \leq n$.

Gröbner bases for $I(X_z)$

Theorem (Marberg–Pawlowski)

For $z \in \mathcal{I}_n^{\mathrm{FPF}}$, a Gröbner basis for $I(X_z)$ is given by the polynomials

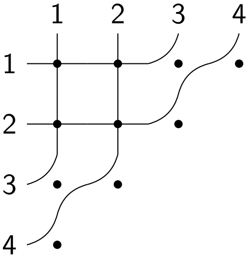
$$g_{AB} := \mathrm{pf} \begin{bmatrix} \mathcal{U}_{BB} & \mathcal{U}_{BA} \\ \mathcal{U}_{AB} & 0 \end{bmatrix}$$

running over an explicit collection of sets $A, B \subseteq [n]$ depending on z.

- If $A = \emptyset$, then $g_{AB} = \mathrm{pf}(\mathcal{U}_{BB})$.
- If $A \cap B = \emptyset$ and $|A| = |B|$, then $g_{AB} = \pm \det(\mathcal{U}_{AB})$ (cf. identity of Brill)

Fpf involution pipe dreams

An fpf involution pipe dream for $z = (1,3)(2,4) = 3412 \in \mathcal{I}_4^{\mathrm{FPF}}$:



An fpf involution pipe dream for $z \in S_n$ is...

- a symmetric tiling of $[n] \times [n]$ with tiles  and 
- where the i^{th} pipe on the left is the $z(i)^{\text{th}}$ pipe on the top...
- and no pair of pipes crosses more than once.

Identify an fpf involution pipe dream with its set of $+$ tiles strictly below the diagonal, e.g. $\{(2,1)\}$ above.

Primary decompositions

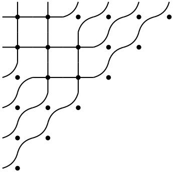
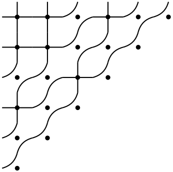
Theorem (Marberg–Pawlowski)

For $z \in \mathcal{I}_n^{\mathrm{FPF}}$, init $I(\mathrm{SSX}_z) = \bigcap_D (u_{ij} : (i,j) \in D)$ *where* D *runs over the set of fpf involution pipe dreams of* z.

That is: the zero locus of init $I(\mathrm{SSX}_z)$ is the union of linear subspaces $\{A \in \mathrm{SSM}_n : A_{ij} = 0 \text{ for } (i,j) \in D\}$.

Example

$z = (1,3)(2,5)(4,6)$ has two fpf involution pipe dreams:

$\{(2,1),(3,2)\} =$  $\{(2,1),(4,1)\} =$ 

- $I(\mathrm{SSX}_{(1,3)(2,5)(4,6)}) = (u_{21}, u_{32}u_{41} - u_{31}u_{42} + u_{43}u_{21})$
- init $I(\mathrm{SSX}_{(1,3)(2,5)(4,6)}) = (u_{21}, u_{32}u_{41}) = (u_{21}, u_{32}) \cap (u_{21}, u_{41})$.

Primary decompositions

Corollaries à la Knutson–Miller:

- Torus-equivariant cohomology class of SSX_z is $\sum_D \prod_{(i,j)\in D}(x_i + x_j)$ where D runs over fpf involution pipe dreams of z.
- Similar formula for equivariant K-theory class in terms of nonreduced fpf involution pipe dreams
- Stanley–Reisner complex of init $I(\mathrm{SSX}_z)$ has facets in bijection with fpf inv. pipe dreams of z, and is shellable.

Proofs

Induct on Bruhat order restricted to $\mathcal{I}_n^{\mathrm{FPF}}$:

- $\mathrm{SSX}_z \cap \{A \in \mathrm{SSM}_n : \operatorname{rank} A_{[p][q]} \leq 0\} = \bigcup_y \mathrm{SSX}_y$ as sets.
- True as schemes? i.e. does

$$I(\mathrm{SSX}_z) + (u_{pq}) = \bigcap_y I(\mathrm{SSX}_y)? \tag{1}$$

- Define J_z as ostensible initial ideal, show using pipe dream combinatorics that $J_z + (u_{pq}) = \bigcap_y J_y$.
- Gives enough control to conclude that $J_z = \operatorname{init} I(\mathrm{SSX}_z)$, that (1) holds, and that $I(\mathrm{SSX}_z)$ equals the ideal generated by appropriate Pfaffians.

Proofs

- This is different from Knutson and Miller's approach, and gives new proofs of many of their results on matrix Schubert varieties.
- Hope: apply this technique to the more difficult setting of symmetric matrix Schubert varieties (related to O_n-orbits on the type A flag variety).