Refined Dubrovin conjecture for coadjoint varieties.

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I Dubrovin's conjecture

X Fano / $\mathbb{C}$, smooth, projective.

Dubrovin's conjecture relates

$$D^b(X) \longleftrightarrow BQH(X)$$

bounded derived category of X — big quantum cohomology of X.

Conjecture:

$D^b(X)$ has a full. exc. collection $\iff$ $BQH(X)$ is gen. semisimple.

II Reminders on $D^b(X)$

- <u>Def:</u> (1) $E \in D^b(X)$ is <u>exceptional</u> if $\mathrm{Hom}^\bullet(E,E) = \mathbb{C}$.

  (2) $E_1 \dots E_k \in D^b(X)$ is an <u>exc. collection</u> if
  - E_i are exceptional.
  - $\mathrm{Hom}^\bullet(E_i, E_j) = 0$ for $i > j$.

- Assume $\mathrm{Pic}(X) = \mathbb{Z}H$, H = ample generator.

  <u>Def:</u> The index of X is

  $$m = \mathrm{Index}(X) \in \mathbb{Z}_{\geq 0} \quad \text{s.t.} \quad -K_X = mH$$

- <u>Def:</u> (1) An exc. collection $E_1, \dots, E_k$ extends to a <u>rectangular Lefschetz collection</u> if the collection

  $$(E_1 \dots E_k, E_1(1), \dots, E_k(1), \dots, E_1(m-1), \dots, E_k(m-1))$$

  is exceptional (here $E_i(j) = E_i \otimes \mathcal{O}_X(j) = E_i \otimes \mathcal{O}_X(jH)$.)

  (2) A Lefschetz collection $(E_i(j))_{\substack{1\leq i\leq k\\ 0\leq j\leq m-1}}$ is full if it generates $D^b(X)$:

  $$D^b(X) = \langle E_1, \dots, E_k, \dots, E_1(m-1), \dots, E_k(m-1)\rangle.$$

  (3) The <u>residual category</u> of a Lefsch. collection $(E_i(j))$ is

  $$\mathcal{R} = \langle E_i(j) \mid i,j\rangle^\perp = \{A \mid \mathrm{Hom}^\bullet(E_i(j), A) = 0 \ \forall i,j\}.$$

  We have

  $$D^b(X) = \langle \mathcal{R}, E_1, \dots, E_k, \dots, E_1(m-1), \dots, E_k(m-1)\rangle$$

III Reminders on $QH(X)$ and $BQH(X)$

- Both are deformation of $(H^*(X), \cup)$.

- Very vague description: same space but deformed product using more parameters: Choose a basis of $H^{even}(X)$

get variables

$$\begin{array}{cccc} 1 & H & \Delta_1 \cdots\cdots & \Delta_N \\ \downarrow & & & \\ & q & t_1 \cdots\cdots & t_N \end{array}$$

$$\Delta_a * \Delta_b = \Delta_a \cup \Delta_b + \sum_{\substack{d>0 \\ i_1, \dots, i_N > 0}} q^d t_1^{i_1} \cdots t_N^{i_N} \underbrace{c_{ab}^c(d, i_1, \dots, i_N)}_{\text{counts rat. curve in } X \text{ of deg } d} \Delta_c$$

$BQH(X) = (H^*(X), *)$, $QH(X) = (H^*(X), *_{t_1 = \cdots = t_N = 0})$

$H^*(X) = (H^*(X), *_{q = t_1 = \cdots = t_N = 0})$.

Theorem: This product is graded, commutative and associative

$\deg q = m = \mathrm{Index}(X)$

$\deg t_i = 1 - \deg \Delta_a$.

IV Examples.

• $X = \mathbb{P}^n$, $m = n+1$.

$D^b(X) = \langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n) \rangle$ Full rect. Lefs. collection.

$H^*(X) = \mathbb{Q}[h]/(h^{n+1})$. nilpotent.

$QH(X) = \mathbb{Q}[h]/(h^{n+1} - q)$ semisimple.

• X = quadric $\subset \mathbb{P}^{n+1}$ $m = n = \dim X$.

$$D^b(X) = \begin{cases} \langle \underbrace{S}_{\text{residual}}, \underbrace{\mathcal{O}_X, \dots, \mathcal{O}_X(n-1)}_{\text{Lefs. coll.}} \rangle & n \text{ odd} \\ \langle \underbrace{S_-, S_+}_{\text{residual}}, \underbrace{\mathcal{O}_X, \dots, \mathcal{O}_X(n-1)}_{\text{Lefs. coll.}} \rangle & n \text{ even.} \end{cases}$$

eigenspaces of $\neq 0$ eigenvalues of h.

$$QH(X) = \begin{cases} \mathbb{Q}[h,q]/(h^{n+1} - hq) & n \text{ odd.} \\ \mathbb{Q}[h,q,\gamma]/(h^{n+1} - hq, h\gamma, \gamma^2 - q) & n \text{ even.} \end{cases}$$

Kernel of h

V Mirror symmetry

X Fano $\xleftrightarrow{\text{HMS}}$ (Y,f) mirror pair $f: Y \to \mathbb{A}^1$.

	X	(Y,f)
algebraic	$D^b(X)$	$MF(Y,f)$
symplectic	$DFuk(X)$	$DFS(Y,f)$.

eq. of categories.

$$DFuk(X) \simeq MF(Y,f) \underset{HH^*}{\Longrightarrow} QH(X) = \text{Milnor Algebra}(f)$$

leads to "equivalences"

$BQH(X)$ semisimple "$\Longleftrightarrow$" f has only simple critical points "$\Longleftrightarrow$" $DFS(Y,f)$ has a full exc. coll "$\Longleftrightarrow$" $D^b(X)$ has a full exc. coll.

VI Refined conjecture (after Kuznetsov - Smirnov)

Look at $QH(X)|_{q=1}$ $\rightsquigarrow$ has a $\mathbb{Z}/m\mathbb{Z}$-grading ($\deg q = m$).

$\rightsquigarrow$ μ_m-action on $\operatorname{Spec} QH(X)$ [μ_m = m^{th} root of 1]

$\rightsquigarrow$ expect $\exists (Y, f)$ mirror with a μ_m-action.

$\rightsquigarrow$ critical pts behave in orbits $\to$ free μ_m-orbits, non free orbits.

Conjecture [Kuznetsov - Smirnov]

X Fano, $m = \operatorname{Index}(X)$ and assume $BQH(X)$ semisimple.

Set k = # free μ_m-orbits in $QH(X)$

(1) $\exists E_1 \dots E_k, \dots, E_1(m-1), \dots, E_k(m-1)$ a Lefschetz collection.

(2) Let R = residual category, then $\exists$ orthogonal decomp°

$$\mathcal{R} = \bigoplus_{\substack{\xi \text{ non free} \\ \text{orbits}}} \mathcal{R}_\xi .$$

(3) If a non-free orbit ξ is a germ of an ADE-sing, then

$$\mathcal{R}_\xi = D^b(\text{Rep. of the corresp ADE-quiver}).$$

VII Examples: coadjoint varieties.

Original example.

Theorem [Cruz-Morales, Kuznetsov, Mellit, P, Smirnov].

- For $X = IG(2,2n)$ = isotropic lines in $\mathbb{C}^{2n}$ for a sym. form.

  $m = 2n-1$

(1) $BQH(X)$ is semisimple.

(2) $QH(X)$ has k free orbits and a unique fat point of type A_{n-1}.

(3) $D^b(X) = \langle \mathcal{R}, E_1, \dots, E_k, \dots, E_1(m-1), \dots, E_k(m-1) \rangle$

and $\mathcal{R} \simeq D^b$ (Rep of quiver A_{n-1}).

Def: (1) G simple, $\mathfrak{g} = \mathrm{Lie}(G)$ then

X <u>adjoint</u> = closed G orbit in $\mathbb{P}(\mathfrak{g})$.

(2) X adjoint is asso. to nodes in the D.D of G.

Take $G^\vee$ = Langlands dual = same diagram with reversed arrow.

X <u>coadjoint</u> = same nodes in the D.D of $G^\vee$.

Ex: (1) G simply laced X adjoint $= X$ coadjoint.

(2) E_6/P_2, E_7/P_1, E_8/P_8

$OG(2,2n)$, $Fl(1,n \mid n+1)$ ...

(3) Type C: $IG(2,2n)$

Type B: Q_{2n-1}

Type F_4: F_4/P_4, Type G_2: Q_5.

Theorem [P-Smirnov]

For X coadjoint $BQH(X)$ is semisimple and $QH(X)$ has k free orbits and a unique fat point of type = Dynkin diagram of <u>short</u> roots.

<u>Thm</u> [Smirnov-Kuznetsov] X coadjoint G of type $ABCD$.

$$D^b(X) = \langle \mathcal{R}, E_1, \dots, E_k, \dots, E_1(n-1), \dots, E_k(n-1) \rangle$$

$$\mathcal{R} = D^b(\text{Rep. of quiver of short roots})$$