Bow varieties

bow

arrows
(a.k.a. homomorphisms)

quiver

- joint work with Yiyan Shou
- learned about branes from Lev Rozansky
- related works with

  Andrey Smirnov
  Alexander Varchenko
  Zijun Zhou
  Andrzej Weber

At the beginning there were (type-A) homogeneous spaces...

$Gr_2 \mathbb{C}^4$

$\mathcal{F}_{2,5,7}$

$\mathcal{F}_{1,2,3,4}$

... and we did Schubert Calculus on them.

Schubert Calculus $\rightarrow$ cotangent Schubert Calculus

(a.k.a. $\hbar$-deformed Schubert Calculus)

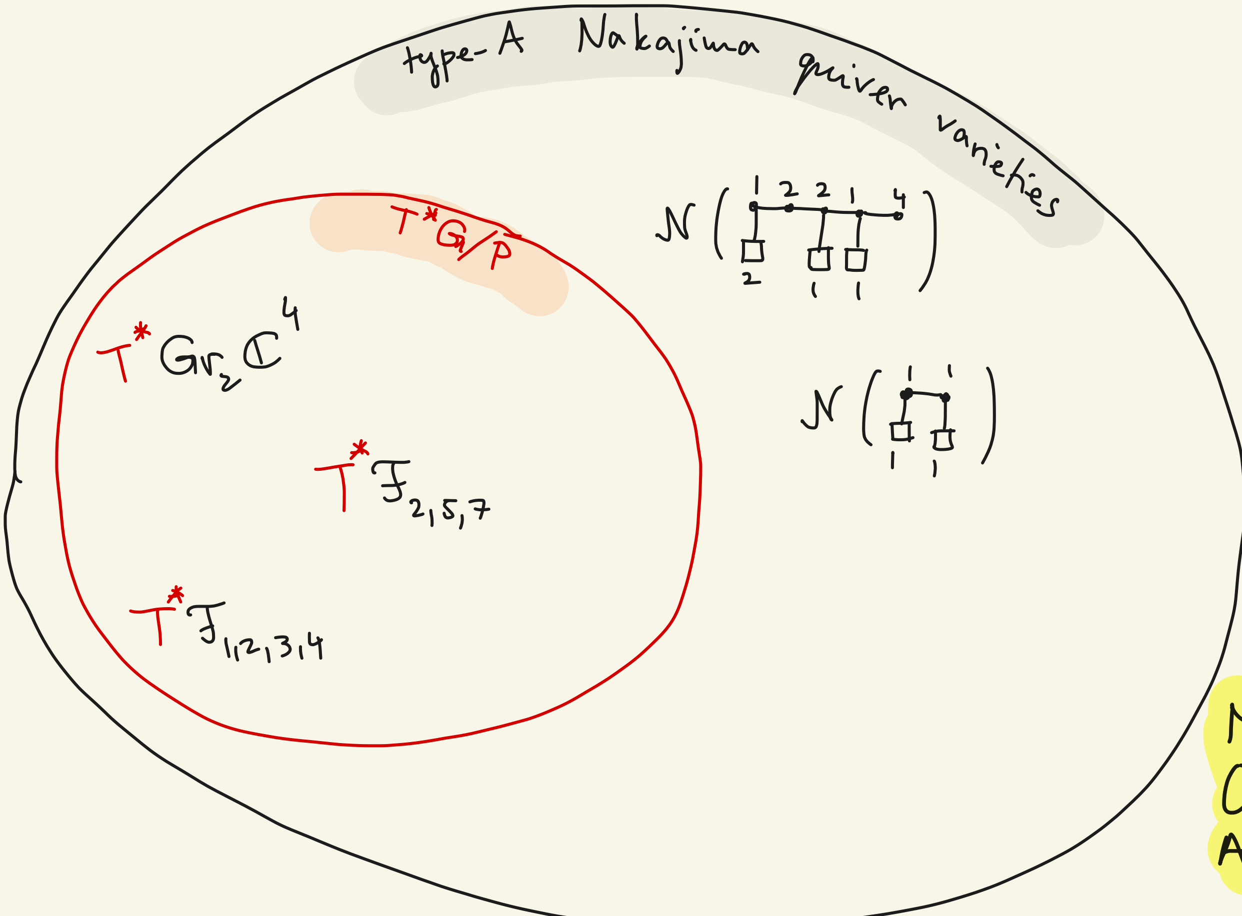
$T^*Gr_2\mathbb{C}^4$

$T^*\mathcal{F}_{2,5,7}$

$T^*\mathcal{F}_{1,2,3,4}$

Historically:

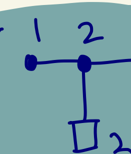
cotangent Sch Calc $\begin{cases} H^*: \text{"Chern-Schwartz-MacPherson classes"} \\ K: \text{"Motivic Chern classes"} \end{cases}$



Maulik–Okounkov
Okounkov
Aganagić-Okounkov

… Schubert classes ("stable envelopes") on $\mathcal{N}$(quiver)

THE COINCIDENCE !!!

$T^*Gr_2\mathbb{C}^4 \longleftrightarrow \mathcal{N}\Big($  $\Big)$

[RSVZ 2020]

intimate relationship between their Schubert Calculus

"$N=4$ $d=3$ mirror symmetry for characteristic classes"

dim = 8

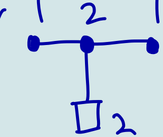
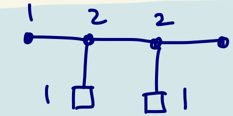
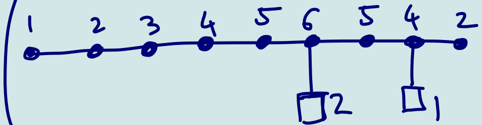
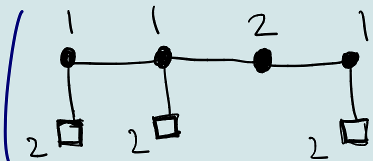
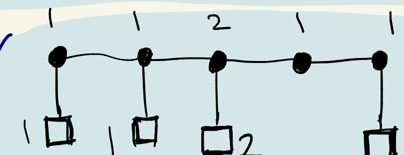
fix pts = 6

T^4 action

dim = 4

fix pts = 6

T^2 action

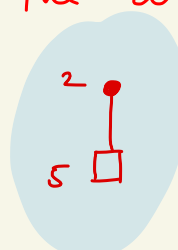
dim				dim	
8	$T^*Gr_2\mathbb{C}^4$	$\longleftrightarrow$	$\mathcal{N}\Big($  $\Big)$	4	[RSVZ]
12	$T^*Gr_2\mathbb{C}^5$	$\longleftrightarrow$	$\mathcal{N}\Big($  $\Big)$	4	
64	$T^*\mathcal{F}_{2,6,10}$	$\longleftrightarrow$	$\mathcal{N}\Big($  $\Big)$	16	
8	$\mathcal{N}\Big($  $\Big)$	$\longleftrightarrow$	$\mathcal{N}\Big($  $\Big)$	10	
	T^*G/B	$\longleftrightarrow$	T^*G^L/B^L		[RW 2020]
32	$T^*\mathcal{F}_{2,5,7}$	$\longleftrightarrow$	$\mathcal{N}\Big($  $\Big)$		

$\boxed{\text{dim}}$

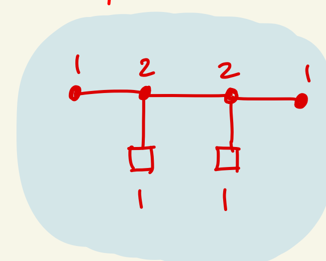
1. What exactly is the relationship between Schubert classes of 3d mirror dual spaces?

2. How to find the 3d mirror dual?

(ie what is the combinatorics that connects

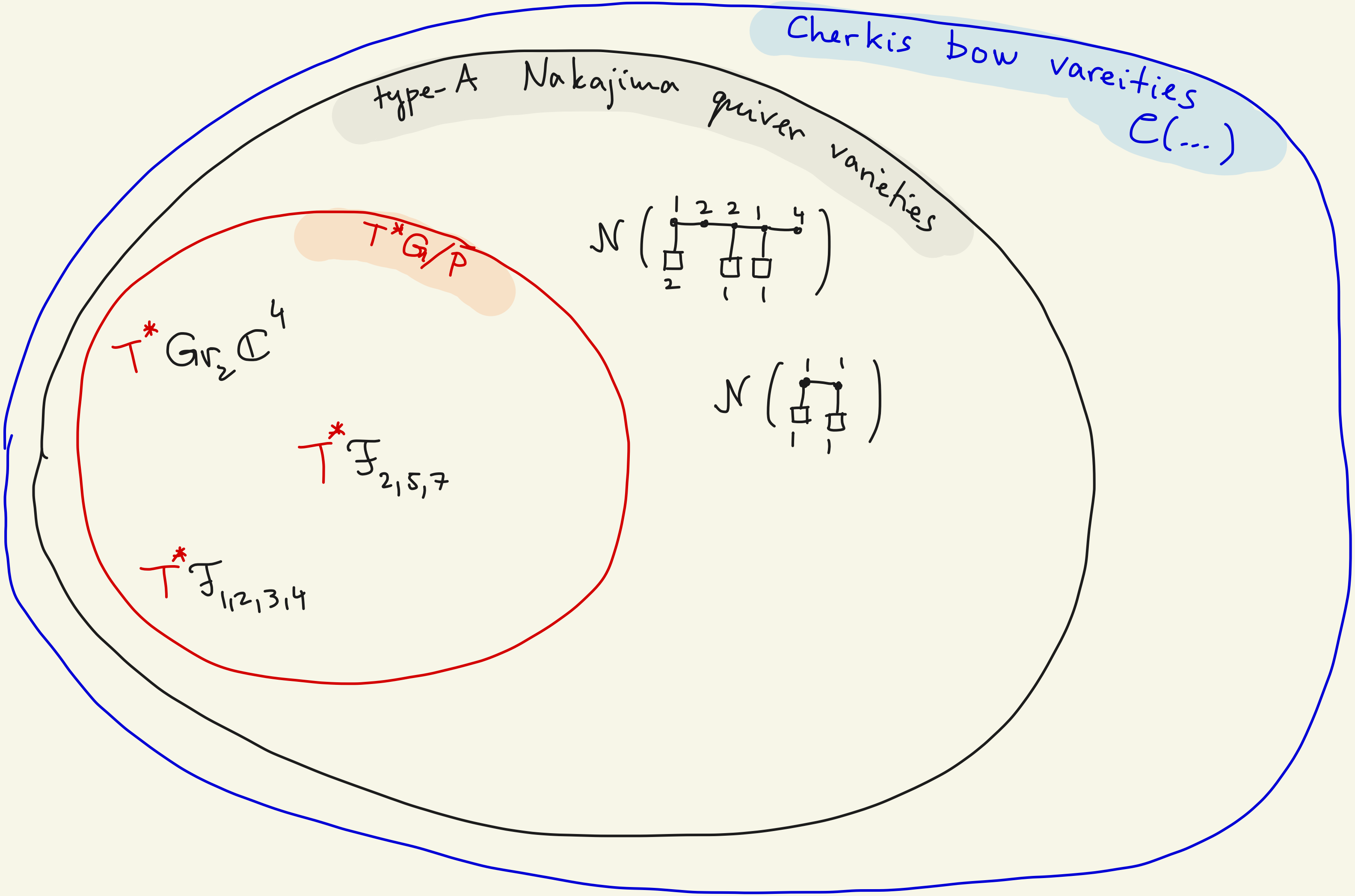


with

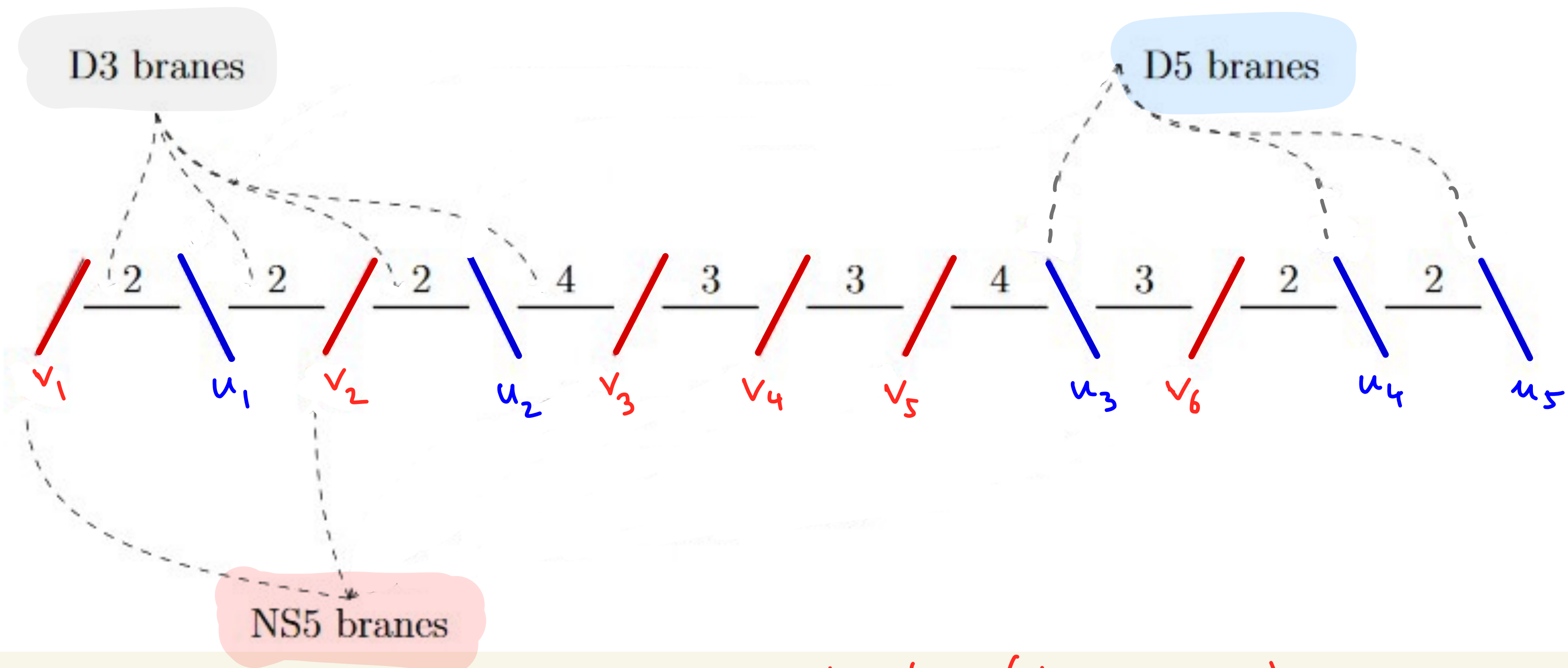


?)

(what is the mirror of $T^*\mathcal{F}_{2,5,7}$?)


Cherkis bow vareities
$\mathcal{C}(\ldots)$
type-A Nakajima quiver varieties
T^*G/P
$T^*Gr_2\mathbb{C}^4$
$T^*\mathcal{F}_{2,5,7}$
$T^*\mathcal{F}_{1,2,3,4}$

Brane diagrams



v_i : Kähler (dynamical) variables

u_i : equivariant variables

brane diagram $\mathcal{D}$ $\longrightarrow$ $C(\mathcal{D})$

tautological bundles, one for each D3 brane

$C(\mathcal{D})$ — D5 branes $\top$

Cherkis bow variety

<u>Cherkis:</u>
moduli space of unitary instantons on multi-Taub-NUT spaces
(key: Nahm's equation)

<u>Nakajima-Takayama</u>
Hamiltonian reduction of representations of certain quivers with relations

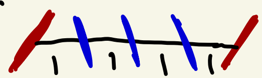
$\sim$

<u>Rozansky-R</u>
"symplectic intersection" of generalized Lagrange varieties

$$\dim(C(D)) = \sum_{U \quad D5} \Big[(d_{u_-}+1)d_{u_-} + (d_{u_+}+1)d_{u^+} \Big]$$

$$+ \sum_{V \quad NS5} 2\, d_{v^+} d_{v^-} \qquad - 2 \sum_{X \quad D3} d_X^2$$

<u>example</u>

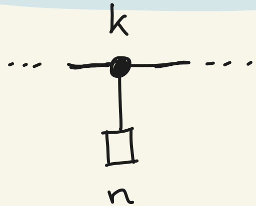
$\dim\Big(C\big($ 

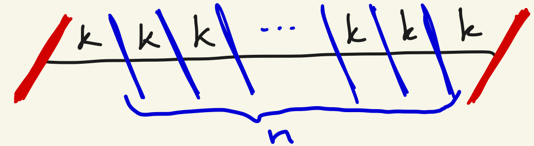
$\big)\Big) = 2\cdot 1 + 2\cdot 1 + 2\cdot 1 + 2\cdot 1 + 2\cdot 1 + 2\cdot 1$

$+ 2\cdot 0\cdot 1 + 2\cdot 1\cdot 0 \quad - 2\left(1^2 + 1^2 + 1^2 + 1^2\right)$

$= 4$

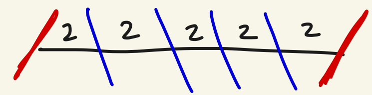
How are $\mathcal{N}(\text{quiver})$ special cases?

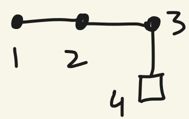
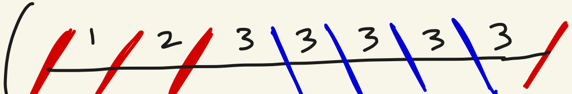


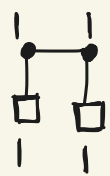
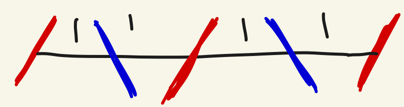


<u>Examples</u>

$T^*\mathbb{P}^1 = \mathcal{N}\Big($  $\Big) = \mathcal{C}\Big($  $\Big)$

$T^*Gr_2\mathbb{C}^4 = \mathcal{N}\Big($  $\Big) = \mathcal{C}\Big($  $\Big)$

$T^*\mathcal{F}_{1,2,3,4} = \mathcal{N}\Big($  $\Big) = \mathcal{C}\Big($  $\Big)$

$\mathcal{N}\Big($  $\Big) = \mathcal{C}\Big($  $\Big)$

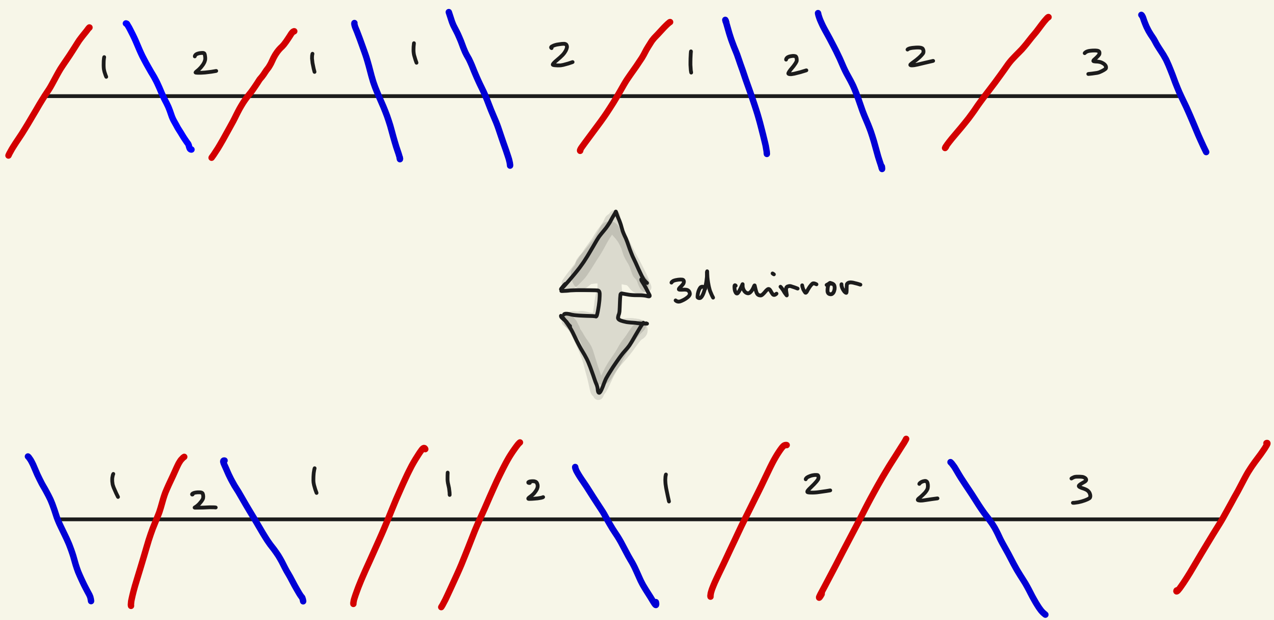
<u>Observe</u>  "cobalanced brane diagram"

There is a lot to say about $C(D)$ spaces.

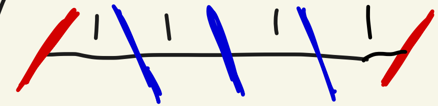
Today:

- 3D mirror symmetry
- Hanany-Witten (HW) transition
- combinatorics of torus fixed points
- Conj 3d mirror symmetry for char classes

3D mirror symmetry for bow varieties:

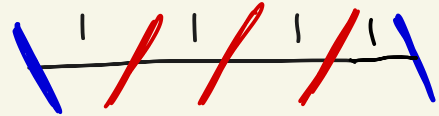


Ex

$$T^*\mathbb{P}^2 = \mathcal{N}\left(\begin{smallmatrix} \overset{1}{\bullet} \\ | \\ \square_3 \end{smallmatrix}\right) = \mathcal{C}\Big($$  $$\Big)$$

dim 4

$\Updownarrow$ 3d mirror

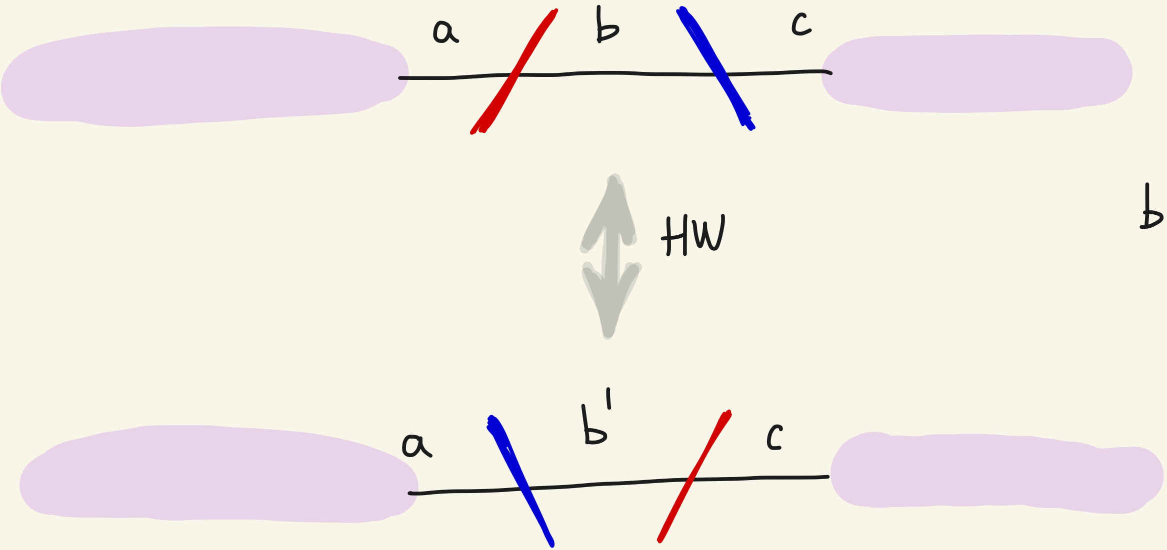
$$\mathcal{C}\Big($$  $$\Big)$$

dim 2

not cobalanced, ie not $\mathcal{N}(\ldots)$

... but ... <to be continued>

Hanany-Witten transition on brane diagrams.

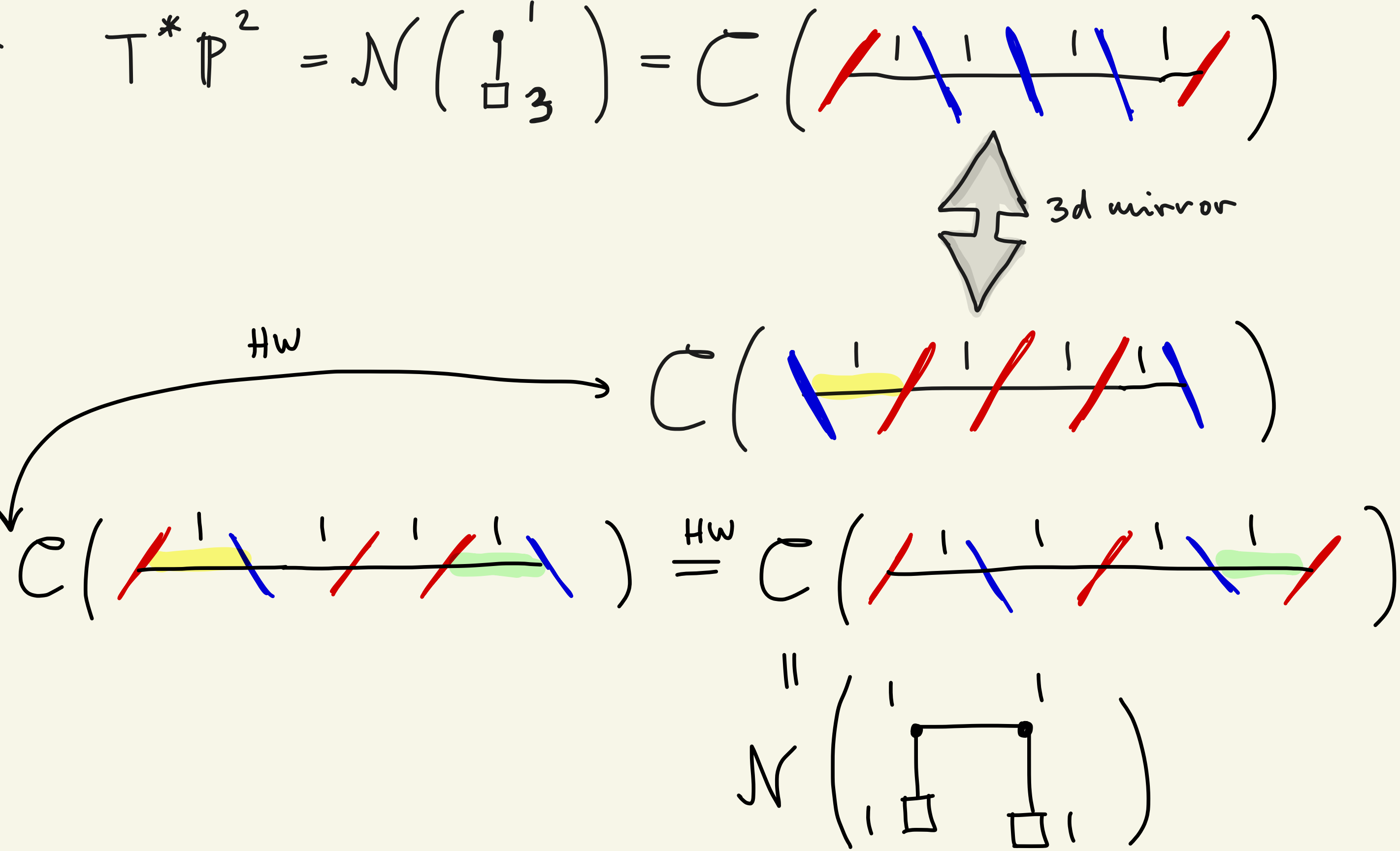


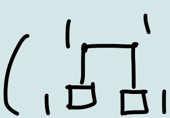
$b + b' = a + c + 1$

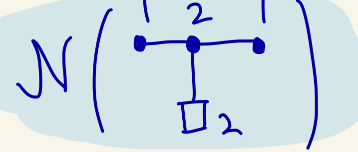
(why? later: "brane charge")

$\underline{\text{Thm}}$ $\quad \mathcal{C}(\mathcal{D}) \cong \mathcal{C}(\, HW(\mathcal{D})\,)$

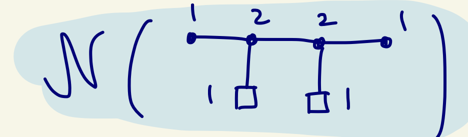
Ex

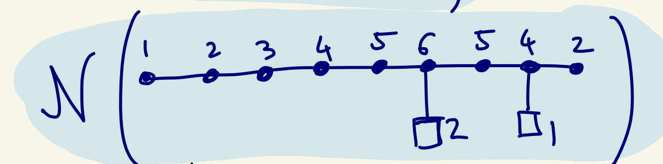


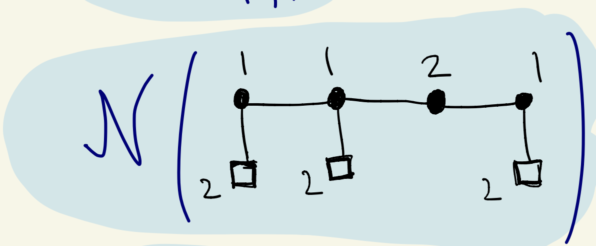
$\Rightarrow$ $T^*\mathbb{P}^2 \xleftrightarrow{\text{3d mirror}} \mathcal{N}($  $)$

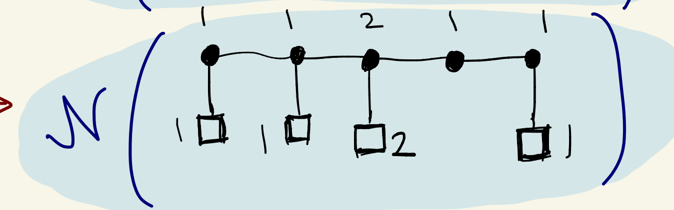
$T^*Gr_2\mathbb{C}^4 \longleftrightarrow$ 

[RSVZ]

$T^*Gr_2\mathbb{C}^5 \longleftrightarrow$ 

$T^*\mathcal{F}_{2,6,10} \longleftrightarrow$ 



$\longleftrightarrow$


$T^*G/B \longleftrightarrow T^*G^L/B^L$

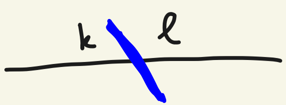
[RW 2020]

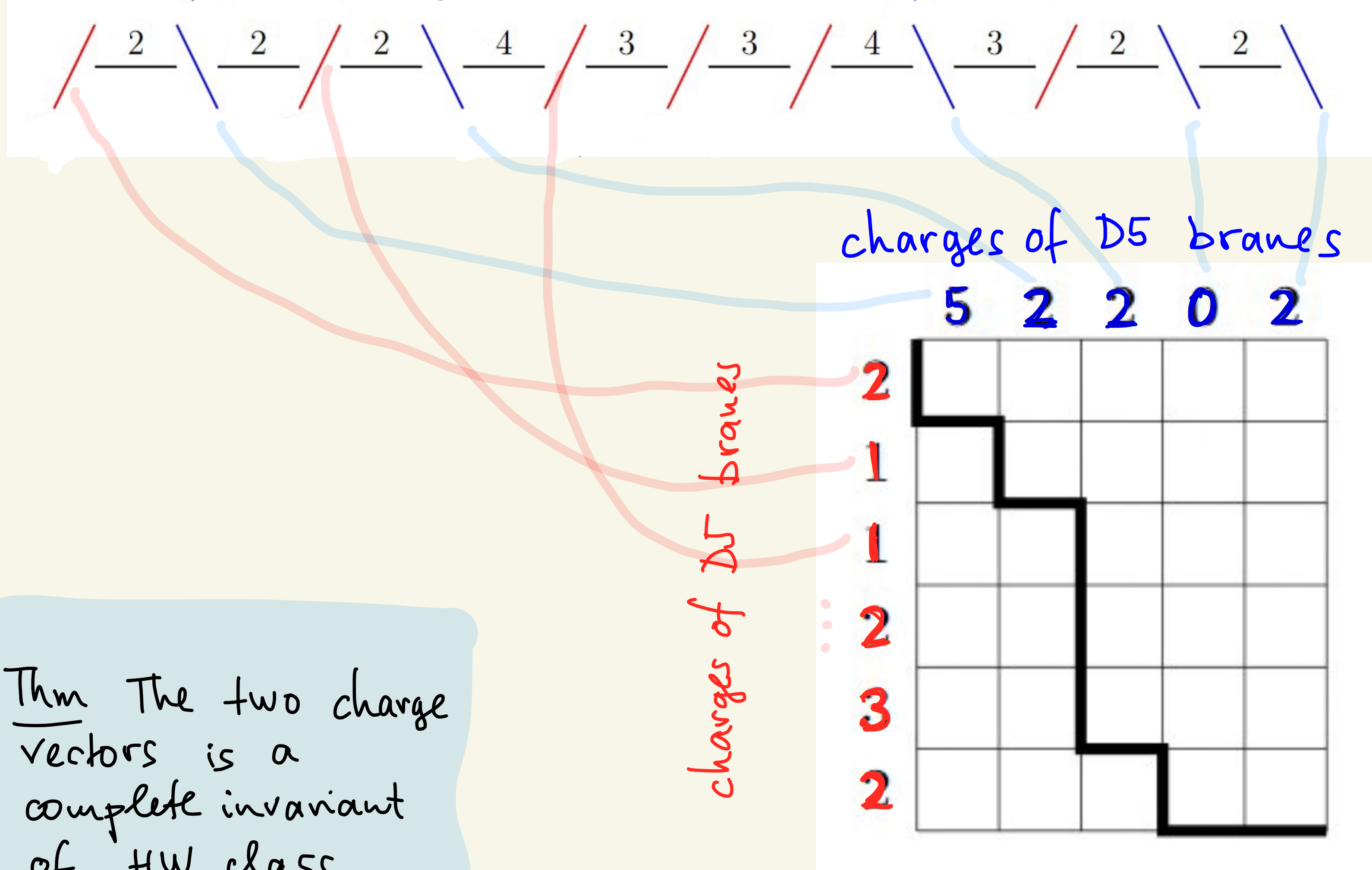
$T^*\mathcal{F}_{2,5,7} \longleftrightarrow$ 

<u>def</u> brane charge

$$\mathrm{charge}\left(\text{NS5 brane}\right) := \ell - k + \#\{\text{D5-branes left of it}\}$$

$$\mathrm{charge}\left(\text{D5 brane}\right) := k - \ell + \#\{\text{NS5-branes right of it}\}$$



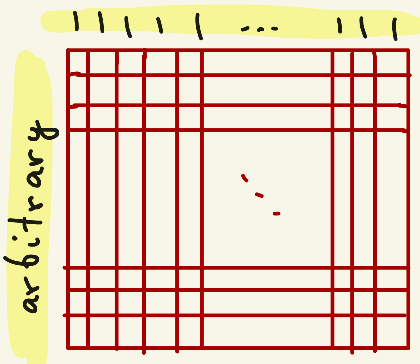


<u>Thm</u> The two charge vectors is a complete invariant of HW class

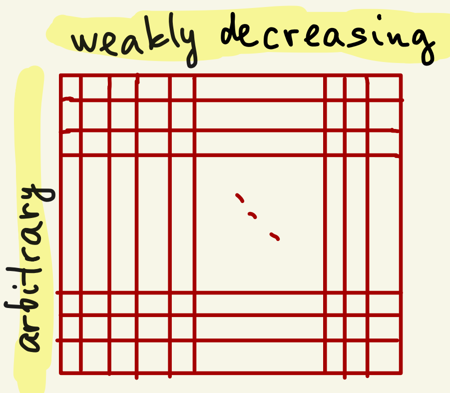
Thm (up to HW transitions)

$$T^*G/P \;\subset\; \mathcal{N}(\text{quiver}) \;\subset\; \mathcal{C}(\text{brane diagram})$$

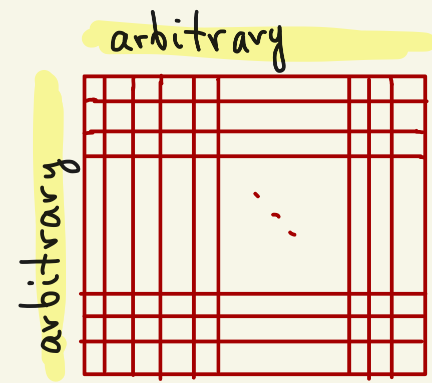
$\updownarrow$



$\updownarrow$

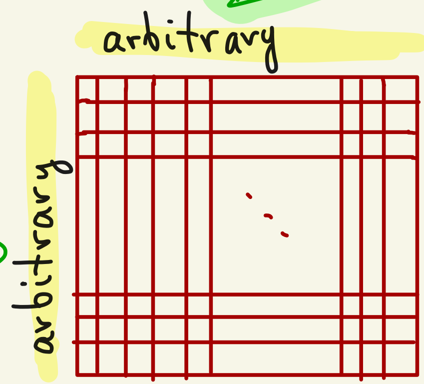


$\updownarrow$



Digression: Okounkov's theory: geometric construction of quantum group actions on H^*_T, K_T, Ell_T.

Expectation: (known for quivers)

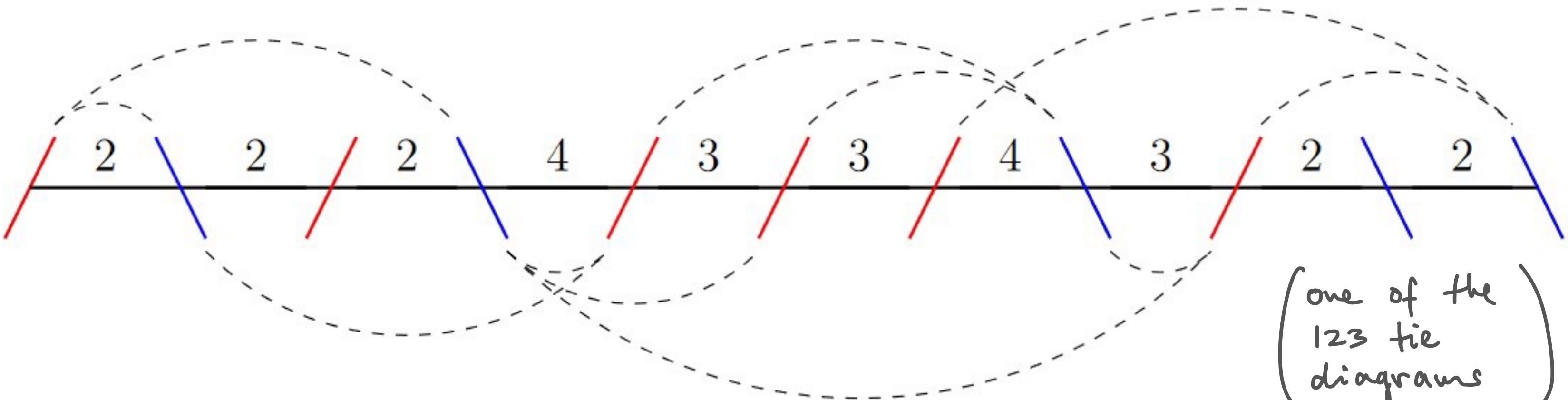


Combinatorial codes for

$\underbrace{\text{TORUS FIXED POINTS}}$

(in physics : "exact vacuums")

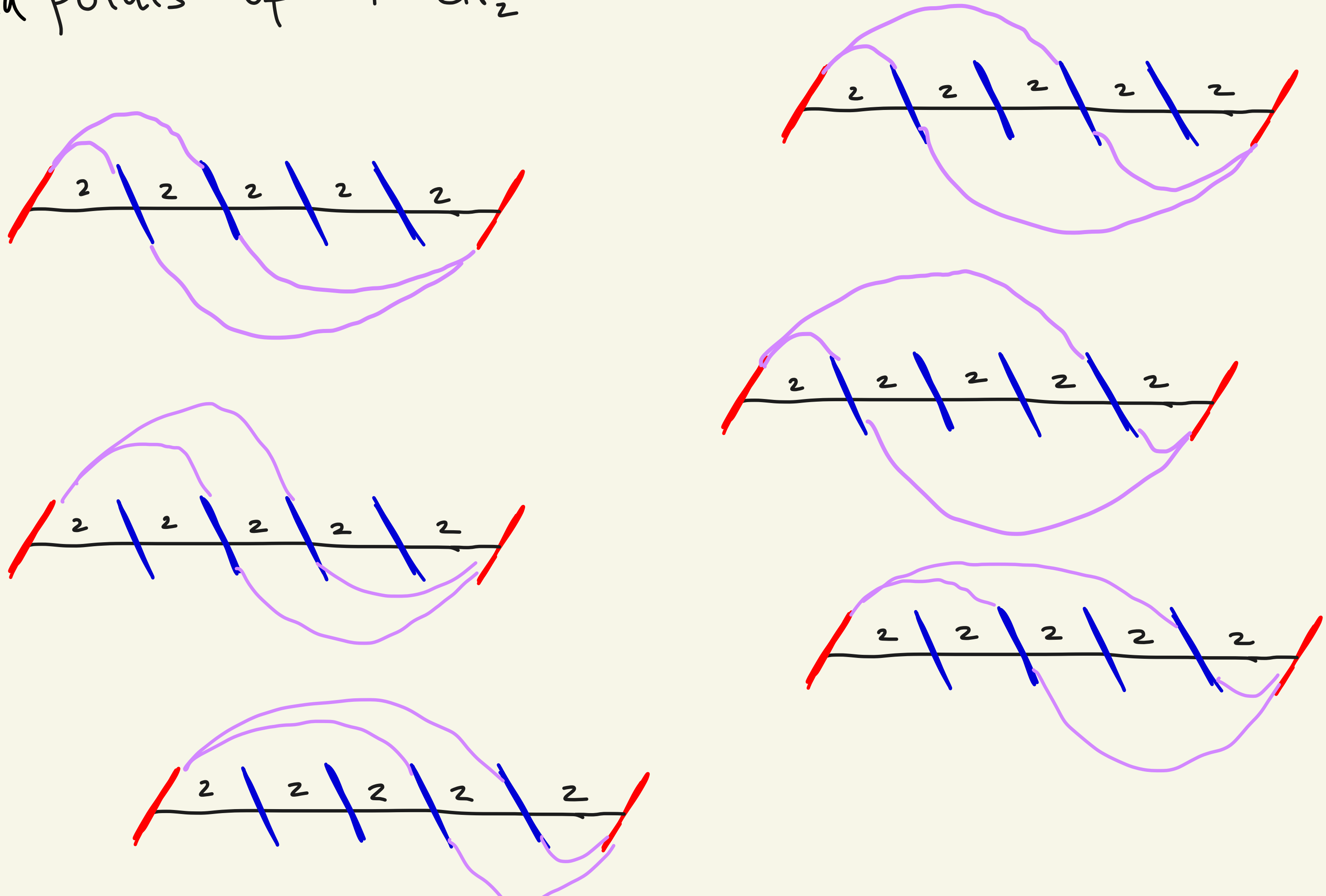
[redacted] AirSpeed EXACT Reach AS3008A Upright Vacuum, Bagless, Allergy Filter, Blue/Black fixed points $\overset{1:1}{\longleftrightarrow}$ tie diagrams

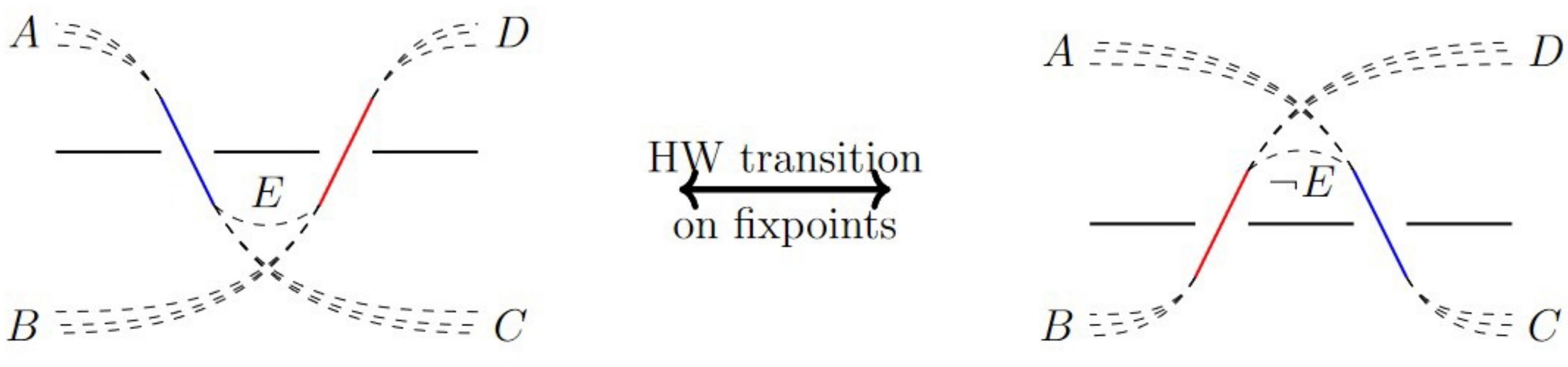


(one of the 123 tie diagrams)

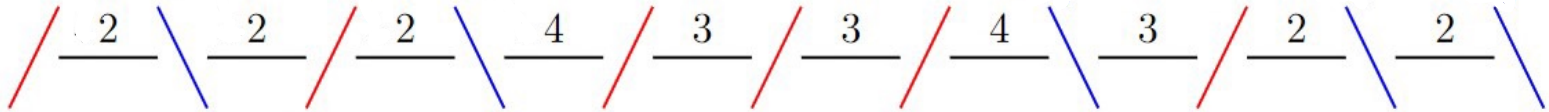
- a tie must connect 5-branes of different kinds
- each D3 brane to be cored as many times as its multiplicity

fixed points of $T^* Gr_2 \mathbb{C}^4$:





$\sim$ Reidemeister-III



binary contingency tables

BCT : 0-1-matrix with row & column sums the charge vectors

Thm

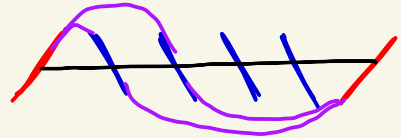
fix pts $\overset{1:1}{\longleftrightarrow}$ BCT's

one of the 123 BCTs

	5	2	2	0	2
2	1	1	0	0	0
1	1	0	0	0	0
1	0	0	1	0	0
2	1	0	1	0	0
3	1	1	0	0	1
2	1	0	0	0	1

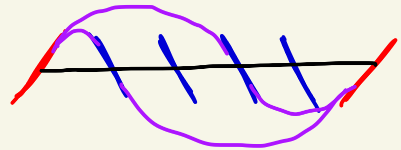
$Gr_2\,\mathbb{C}^4$

$\{1,2\}$

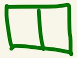


$$\begin{array}{c|cccc} & 1 & 1 & 1 & 1 \\ \hline 2 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 1 \end{array}$$

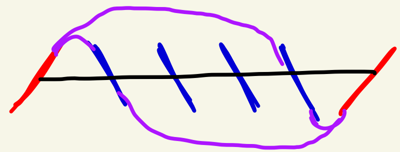
$\{1,3\}$



$$\begin{array}{c|cccc} & 1 & 1 & 1 & 1 \\ \hline 2 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 \end{array}$$

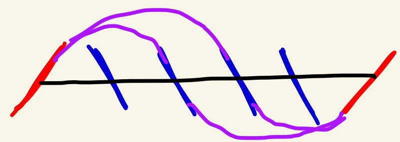


$\{1,4\}$

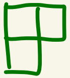


$$\begin{array}{c|cccc} & 1 & 1 & 1 & 1 \\ \hline 2 & 1 & 0 & 0 & 1 \\ 2 & 0 & 1 & 1 & 0 \end{array}$$

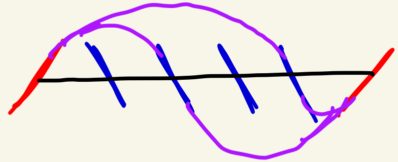
$\{2,3\}$



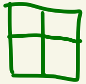
$$\begin{array}{c|cccc} & 1 & 1 & 1 & 1 \\ \hline 2 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \end{array}$$



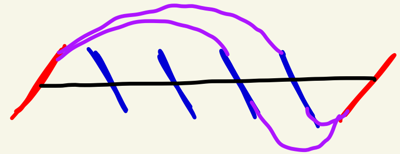
$\{2,4\}$



$$\begin{array}{c|cccc} & 1 & 1 & 1 & 1 \\ \hline 2 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 0 \end{array}$$



$\{3,4\}$



$$\begin{array}{c|cccc} & 1 & 1 & 1 & 1 \\ \hline 2 & 0 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 0 \end{array}$$

So far:

- $\mathcal{C}($ brane diagram $)$

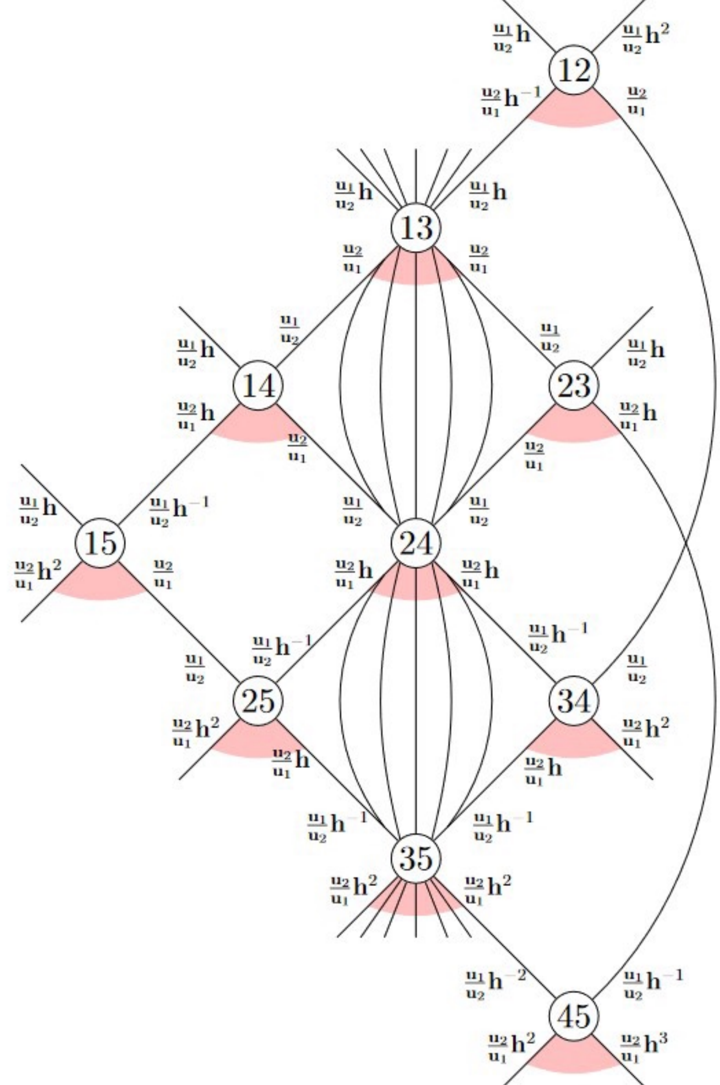
  operations : - 3d mirror

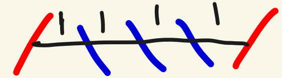
  - HW transition

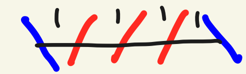
  brane charge

  combinatorics of torus fixed pts.

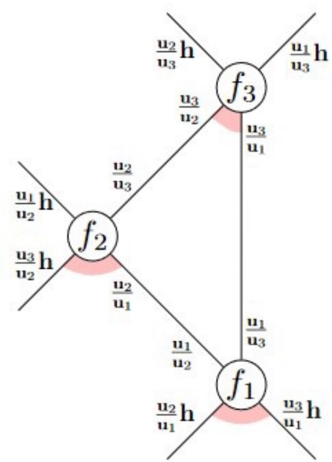
3d mirror of $T^*Gr_2\mathbb{C}^5$

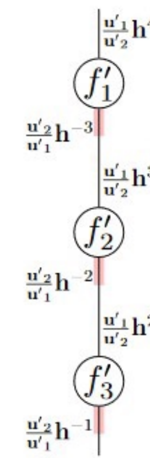




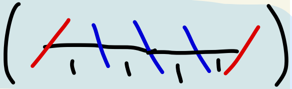


$T^*\mathbb{P}^2$ $\longleftrightarrow$ its mirror

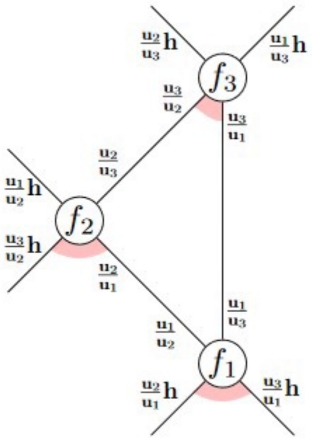




$T^*\mathbb{P}^2 = \mathcal{N}\Big($  $\Big)$

$= \mathcal{C}\Big($  $\Big)$

	f_1	f_2	f_3
f_1	$\theta(\frac{u_1}{u_2})\theta(\frac{u_1}{u_3})\theta(\frac{v_2}{v_1}h^4)$	0	0
f_2	$\theta(h)\theta(\frac{u_1}{u_3})\theta(\frac{u_2v_2}{u_1v_1}h^3)$	$\theta(\frac{u_1}{u_2}h)\theta(\frac{u_2}{u_3})\theta(\frac{v_2}{v_1}h^3)$	0
f_3	$\theta(h)\theta(\frac{u_2}{u_1}h)\theta(\frac{u_3v_2}{u_1v_1}h^2)$	$\theta(h)\theta(\frac{u_1}{u_2}h)\theta(\frac{u_3v_2}{u_2v_1}h^2)$	$\theta(\frac{u_2}{u_3}h)\theta(\frac{u_1}{u_3}h)\theta(\frac{v_2}{v_1}h^2)$



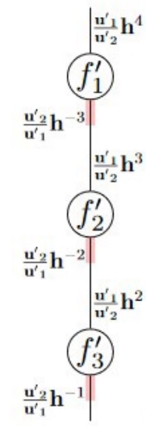
Fact: same after

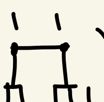
$\pm(\ )^T$

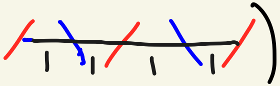
$u_{\cdot} \longleftrightarrow v_{\cdot}$

$h \longleftrightarrow h^{-1}$

In general: conjecture



$\mathcal{N}\Big($  $\Big) =$

$\mathcal{C}\Big($  $\Big)$

	f'_1	f'_2	f'_3
f'_1	$\theta(\frac{u'_1}{u'_2}h^4)\theta(\frac{v'_2}{v'_1})\theta(\frac{v'_3}{v'_1})$	$\theta(h)\theta(\frac{v'_3}{v'_1})\theta(\frac{v'_2u'_2}{v'_1u'_1}h^{-3})$	$\theta(h)\theta(\frac{v'_2}{v'_1}h^{-1})\theta(\frac{v'_3u'_2}{v'_1u'_1}h^{-2})$
f'_2	0	$\theta(\frac{u'_1}{u'_2}h^3)\theta(\frac{v'_2}{v'_1}h)\theta(\frac{v'_3}{v'_2})$	$\theta(h)\theta(\frac{v'_2}{v'_1}h)\theta(\frac{v'_3u'_2}{v'_2u'_1}h^{-2})$
f'_3	0	0	$\theta(\frac{u'_1}{u'_2}h^2)\theta(\frac{v'_3}{v'_2}h)\theta(\frac{v'_3}{v'_1}h)$

$$\left.\begin{array}{l} x_1+x_2+x_3=0 \\ y_1+y_2+y_3=0 \end{array}\right\} \Rightarrow \frac{1}{x_1x_2}+\frac{1}{x_2x_3}+\frac{1}{x_3x_1}=\frac{1}{y_1y_2}+\frac{1}{y_2y_3}+\frac{1}{y_3y_1}$$

H_T^*

$\uparrow$ rational limit ($\sin x \sim x$)

$x_1+x_2+x_3=0,\quad y_1+y_2+y_3=0 \Rightarrow$

$$\cot(x_1)\cot(x_2)+\cot(x_2)\cot(x_3)+\cot(x_3)\cot(x_1)=\cot(y_1)\cot(y_2)+\cot(y_2)\cot(y_3)+\cot(y_3)\cot(y_1)$$

K_T

$\uparrow$ trigonometric limit ($q \to 1$)

$$\left.\begin{array}{l} x_1x_2x_3=1 \\ y_1y_2y_3=1 \end{array}\right\} \Rightarrow \delta(x_1,y_2)\,\delta\!\left(x_2,\tfrac{1}{y_1}\right)+\delta(x_2,y_3)\,\delta\!\left(x_3,\tfrac{1}{y_2}\right)+\delta(x_3,y_1)\,\delta\!\left(x_1,\tfrac{1}{y_3}\right)=0$$

Ell_T

Thank you !