Differential operators for Schur and Schubert polynomials

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Part 1: Schubert

Schubert Polynomials

The divided differences operators is given by

$$\partial_i f := \frac{f - s_i f}{x_i - x_{i+1}}.$$

Definition

For a permutation $w_0 = (n, n-1, \ldots, 1) \in S_n$, we define its Schubert polynomial as

$$\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1 \in \mathbb{Q}[x_1, x_2, \ldots].$$

For a permutation $w \in S_n$,

$$\partial_i \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{ws_i} & \text{if } \ell(ws_i) = \ell(w) - 1 \\ 0 & \text{if } \ell(ws_i) = \ell(w) + 1. \end{cases}$$

Definition

Given a reduced decomposition $h = (h_1, h_2, \ldots, h_{\ell(w)})$. Let $C(h)$ be the set of all $\ell(w)$-tupels $(\alpha_1, \ldots, \alpha_{\ell(w)})$ of positive integers such that

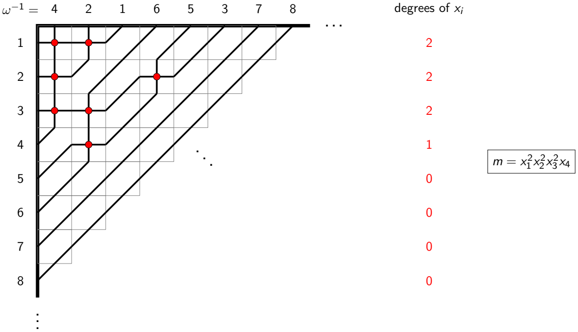
- $1 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{\ell(w)}$;
- $\alpha_j \leq h_j$;
- $\alpha_j < \alpha_{j+1}$ if $h_j < h_{j+1}$.

Theorem (Billey-Jockusch-Stanley, Fomin-Stanley)

For any permutation $w \in S_{\mathbb{N}}$, its Schubert polynomial is given by

$$\mathfrak{S}_w = \sum_{h \in \mathcal{R}(w)} \sum_{\alpha \in C(h)} x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_\ell}.$$

RC graphs/ Pipe dream



Proposition (Fomin-Kirillov)

For any permutation $w \in S_{\mathbb{N}}$, its Schubert polynomial is given by

$$\mathfrak{S}_w = \sum_{g \in \mathcal{RC}(w)} m(g).$$

Theorem

There are unique constants $c_{u,v}^{w}, u, v, w \in S_{\mathbb{N}}$ *such that*

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_{w \in S_{\mathbb{N}}} c_{u,v}^{w} \mathfrak{S}_w.$$

Furthermore, $c_{u,v}^{w}, u, v, w \in S_{\mathbb{N}}$ *are non-negative integers.*

Problem

Give a combinatorial interpretation of $c_{u,v}^{w}$.

Operator ∇

$$\nabla := \sum_{i\in\mathbb{N}} \frac{\partial}{\partial x_i}$$

Theorem (Hamaker-Pechenik-Speyer-Weigandt)

For any $u \in S_{\mathbb{N}}$,

$$\nabla \mathfrak{S}_u = \sum_{k\in\mathbb{N}:\ \ell(s_k u):=\ell(u)-1} k\mathfrak{S}_{s_k u}.$$

Stabilities

Let τ be a shift defined by

$$\tau w(i+1) = w(i) + 1, \ i \in \mathbb{Z},$$

where $w \in S_{\mathbb{Z}}$ is a permutation of $\mathbb{Z}$ fixing all but finitely many elements.

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Stanley symmetric function for $w \in S_{\mathbb{N}}$ is given by

$$\mathcal{F}_w(x_1, x_2, \ldots) := \lim_{k \to +\infty} \mathfrak{S}_{\tau^k w}(x_1, x_2, \ldots) \in \Lambda[x_i, i \geq 1].$$

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Back stable polynomial for $w \in S_{\mathbb{Z}}$ is given by

$$\overleftarrow{\mathfrak{S}}_w(x_i, i \in \mathbb{Z}) := \lim_{k \to +\infty} \mathfrak{S}_{\tau^k w}(x_{1-k}, x_{2-k}, \ldots) \in \Lambda[x_i, i \leq 0] \oplus \mathbb{Q}[x_i, i \in \mathbb{Z}].$$

Theorem (Edelman-Greene)

$$\mathcal{F}_w(x_1, x_2, \ldots) = a_{w,\lambda} s_\lambda(x_1, x_2, \ldots),$$

where $a_{w,\lambda}$ are non-negative.

Theorem (Lam-Lee-Shimozono)

There are unique constants $c_{u,v}^{w}, u, v, w \in S_{\mathbb{Z}}$ *such that*

$$\overleftarrow{\mathfrak{S}}_u \overleftarrow{\mathfrak{S}}_v = \sum_{w \in S_{\mathbb{Z}}} c_{u,v}^{w} \overleftarrow{\mathfrak{S}}_w.$$

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Theorem

Given a pair of permutations $u, v \in S_{\mathbb{Z}}$, the following holds:

$$\binom{\ell(u)+\ell(v)}{\ell(v)} |\mathcal{R}(u)||\mathcal{R}(v)| = \sum_{w \in S_{\mathbb{Z}}} c_{u,v}^w \, |\mathcal{R}(w)|,$$

where $\mathcal{R}(u)$ is the set of reduced words of u.

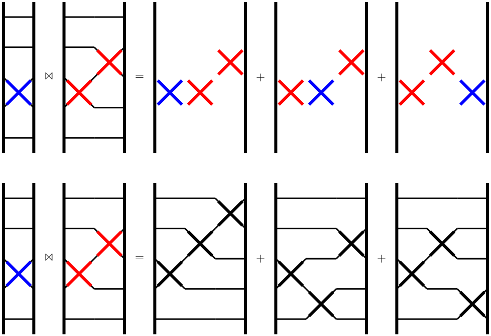

Figure: Merge of reduced decompositions,

$$\overleftarrow{\mathfrak{S}}_{(01324)}\overleftarrow{\mathfrak{S}}_{(02314)} = \overleftarrow{\mathfrak{S}}_{(12304)} + \overleftarrow{\mathfrak{S}}_{(02413)}.$$

Operator ξ on $\overleftarrow{\mathfrak{S}}$

Define ξ as

$$\xi(f) := \sum_{\gamma\in\mathbb{Z}_{\geq 0}^{\mathbb{Z}}} \big(\lim_{k\to-\infty} \text{coef. of } x^\gamma x_k \text{ in } f\big)\cdot x^\gamma = \lim_{k\to-\infty} \frac{\partial f}{\partial x_k}.$$

For Back stable Schubert polynomials, we have

$$\xi \overleftarrow{\mathfrak{S}}_u = \sum_{k:\ \ell(s_k u)=\ell(u)-1} \overleftarrow{\mathfrak{S}}_{s_k u}.$$

Operators ξ and ∇ on $\overleftarrow{\mathfrak{S}}$

$$\xi \overleftarrow{\mathfrak{S}}_u := \sum_{k:\ \ell(s_k u) = \ell(u) - 1} \overleftarrow{\mathfrak{S}}_{s_k u};$$

$$\nabla \overleftarrow{\mathfrak{S}}_u := \sum_{k:\ \ell(s_k u) := \ell(u) - 1} k \overleftarrow{\mathfrak{S}}_{s_k u}.$$

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$$\nabla \overleftarrow{\mathfrak{S}}_u := \sum_{k:\ \ell(s_k u):=\ell(u)-1} k \overleftarrow{\mathfrak{S}}_{s_k u}.$$

Proposition (N.)

For any $u, v \in S_{\mathbb{Z}}$, we have

$$\xi(\overleftarrow{\mathfrak{S}}_u \overleftarrow{\mathfrak{S}}_v) = (\xi \overleftarrow{\mathfrak{S}}_u)\overleftarrow{\mathfrak{S}}_v + \overleftarrow{\mathfrak{S}}_u(\xi \overleftarrow{\mathfrak{S}}_v);$$

$$\nabla(\overleftarrow{\mathfrak{S}}_u \overleftarrow{\mathfrak{S}}_v) = (\nabla \overleftarrow{\mathfrak{S}}_u)\overleftarrow{\mathfrak{S}}_v + \overleftarrow{\mathfrak{S}}_u(\nabla \overleftarrow{\mathfrak{S}}_v).$$

Theorem (N.)

If an operator ζ satisfies:

- $\zeta \overleftarrow{\mathfrak{S}}_u = \sum_{k:\ \ell(s_k u)=\ell(u)-1} b_{u,k} \overleftarrow{\mathfrak{S}}_{s_k u},\ \ b_{u,k} \in \mathbb{Q};$
- $\zeta(\overleftarrow{\mathfrak{S}}_u \overleftarrow{\mathfrak{S}}_v) = (\zeta \overleftarrow{\mathfrak{S}}_u)\overleftarrow{\mathfrak{S}}_v + \overleftarrow{\mathfrak{S}}_u(\zeta \overleftarrow{\mathfrak{S}}_v),$

then ζ is a linear combination of ξ and ∇.

Define the vector space $\mathbb{Q}S_{\mathbb{Z}}$ as formal finite sums of permutations with rational coefficients, i.e.,

$$\mathbb{Q}S_{\mathbb{Z}} := \left\{ \sum_{i=1}^{k} a_i w^{(i)} : \ k \in \mathbb{N}, \ a_i \in \mathbb{Q}, \ w^{(i)} \in \mathbb{Q}S_{\mathbb{Z}} \right\}.$$

Main theorem (weak form)

A *descent* of $u \in S_{\mathbb{Z}}$ is a position $k \in \mathbb{Z}$ with $u(k) > u(k+1)$.

Theorem (N.)

Let $f : \mathbb{Q}S_{\mathbb{Z}} \times \mathbb{Q}S_{\mathbb{Z}} \to \mathbb{Q}S_{\mathbb{Z}},\ f(u,v) = \sum_{w \in \mathbb{Q}S_{\mathbb{Z}}} b^w_{u,v} w$ be a linear map, such that

1. $b^w_{u,v} = 0$ *if* $\ell(w) \neq \ell(u) + \ell(v)$;
2. $b^w_{u,v} = 0$ *if k is a descent of u and* $w(a) \leq k$ *for all* $a \leq k$;
3. $f(id, v) = v$;
4. $\xi f(u,v) = f(\xi u, v) + f(u, \xi v)$;
5. $\nabla f(u,v) = f(\nabla u, v) + f(u, \nabla v)$.

Then $b^w_{u,v} = c^w_{u,v}$ *for all* $u, v, w \in S_{\mathbb{Z}}$.

Main theorem (weak form; symmetric)

Theorem (N.)

Let $f : \mathbb{Q}S_{\mathbb{Z}} \times \mathbb{Q}S_{\mathbb{Z}} \to \mathbb{Q}S_{\mathbb{Z}},\ f(u,v) = \sum_{w \in \mathbb{Q}S_{\mathbb{Z}}} b^w_{u,v} w$ *be a linear map, such that*

1. $b^w_{u,v} = 0$ *if* $\ell(w) \neq \ell(u) + \ell(v)$;
2. $b^w_{u,v} = 0$ *if* k *is a descent of* u *or* v *and* $w(a) \leq k$ *for all* $a \leq k$;
3. $f(id, id) = id$;
4. $\xi f(u,v) = f(\xi u, v) + f(u, \xi v)$;
5. $\nabla f(u,v) = f(\nabla u, v) + f(u, \nabla v)$.

Then $b^w_{u,v} = c^w_{u,v}$ *for all* $u, v, w \in S_{\mathbb{Z}}$.

Main theorem (weak form; positive)

Theorem (N.)

Let $f : \mathbb{Q}S_\mathbb{Z} \times \mathbb{Q}S_\mathbb{Z} \to \mathbb{Q}S_\mathbb{Z}$, $f(u, v) = \sum_{w \in \mathbb{Q}S_\mathbb{Z}} b^w_{u,v} w$ *be a linear map, such that*

1. $b^w_{u,v} = 0$ *if* $\ell(w) \neq \ell(u) + \ell(v)$ *;*
2. $b^w_{u,v} \geq 0$*;*
3. $f(id, id) = id$*;*
4. $\xi f(u, v) = f(\xi u, v) + f(u, \xi v)$*;*
5. $\nabla f(u, v) = f(\nabla u, v) + f(u, \nabla v)$*.*

Then $b^w_{u,v} = c^w_{u,v}$ *for all* $u, v, w \in S_\mathbb{Z}$*.*

Remark

My proof of this theorem is different from proofs of the previous two theorems.

Bosonic operators

Define the sequence of *bosonic* operators

- $\rho^{(1)} := \xi$;
- $\rho^{(k+1)} := \frac{[\rho^{(k)}, \nabla]}{k} = \frac{\rho^{(k)} \cdot \nabla - \nabla \cdot \rho^{(k)}}{k}$.

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Proposition

For any $k \in \mathbb{N}$ and $u, v \in S_{\mathbb{Z}}$, we have

$$\rho^{(k)}(\overleftarrow{\mathfrak{S}}_u \overleftarrow{\mathfrak{S}}_v) = (\rho^{(k)}\overleftarrow{\mathfrak{S}}_u)\overleftarrow{\mathfrak{S}}_v + \overleftarrow{\mathfrak{S}}_u(\rho^{(k)}\overleftarrow{\mathfrak{S}}_v).$$

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Theorem (N.)

Operators $\rho^{(k)}, k \in \mathbb{N}$ commute pairwise.

For a partition λ we define operator ξ^λ as

$$\xi^\lambda := \sum_\mu \frac{\chi^\lambda_\mu}{z_\mu} \rho^{(\mu_1)} \dots \rho^{(\mu_k)}.$$

Proposition

For any $u, v \in S_\mathbb{Z}$ and λ,

$$\xi^\lambda(\overleftarrow{\mathfrak{S}}_u \overleftarrow{\mathfrak{S}}_v) = \sum_{\mu,\nu} c^\lambda_{\mu,\nu} (\xi^\mu \overleftarrow{\mathfrak{S}}_u)(\xi^\nu \overleftarrow{\mathfrak{S}}_v),$$

where $c^\lambda_{\mu,\nu}$ are Littlewood-Richardson coefficients.

Theorem (N.)

For a permutation w and a partition λ, we have

$$\xi^{\lambda}\overleftarrow{\mathfrak{S}}_w = \sum_{\substack{\ell(u)=|\lambda| \\ \ell(u^{-1}w)=\ell(w)-|\lambda|}} a_{\lambda,u}\overleftarrow{\mathfrak{S}}_{u^{-1}w},$$

where $a_{\lambda,u}$ are coefficients in the expressions of Stanley symmetric functions in terms of Schur functions.

Main theorem

Theorem (N.)

Let $f : \mathbb{Q}S_{\mathbb{Z}} \times \mathbb{Q}S_{\mathbb{Z}} \to \mathbb{Q}S_{\mathbb{Z}},\ f(u,v) = \sum_{w \in \mathbb{Q}S_{\mathbb{Z}}} b^w_{u,v} w$ *be a linear map, such that*

1. $b^w_{u,v} = 0$ *if* $\ell(w) \neq \ell(u) + \ell(v)$ *;*
2. $b^w_{u,v} = 0$ *if* k *is a descent of* u *and* $w(a) \leq k$ *for all* $a \leq k$*;*
3. $f(id, v) = v$*;*
4. *for any* $d \in \mathbb{N}$, $\xi^{(d)} f(u,v) = \sum_{i=0}^{d} f(\xi^{(i)} u, \xi^{(d-i)} v)$.

Then $b^w_{u,v} = c^w_{u,v}$ *for all* $u, v, w \in S_{\mathbb{Z}}$.

Remark

We can replace conditions (2) and (3) with symmetric or positive conditions.

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Then $b^w_{u,v} = c^w_{u,v}$ *for all* $u, v, w \in S_{\mathbb{Z}}$.

Remark

We can replace conditions (2) and (3) with symmetric or positive conditions.

Part 2: Schur

A permutation is a Grassmannian permutation if and only if it has at most one descent.

$$\lambda(w) = (w_k - k, w_{k-1} - k + 1, w_{k-2} - k + 2, \ldots).$$

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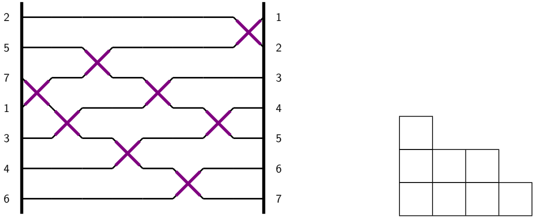


A reduced decomposition of $(2571346) \in S_{\mathbb{Z}}$ and the corresponding Young diagram $(4, 3, 1)$.

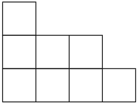
A descent of $w \in S_{\mathbb{Z}}$ is a position $k \in Z$ with $w(k) > w(k+1)$.
A permutation is a Grassmannian permutation if and only if it has at most one descent.

$$\lambda(w) = (w_k - k, w_{k-1} - k + 1, w_{k-2} - k + 2, \ldots).$$

Theorem

$$\mathfrak{S}_w = s_{\lambda(w)}(x_i,\ i \leq k).$$

We denote by $\mathcal{Y}$ the set of Young diagrams (partitions), i.e., $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathcal{Y}$ s.t. $\lambda_1 \geq \ldots \geq \lambda_k \geq 0$, $\lambda_i \in \mathbb{Z}_{\geq 0}$. For example,

$(4, 3, 1) \;=$ 

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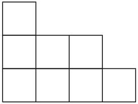
$(4, 3, 1) \quad =$ 

Define the vector space $\mathbb{Q}\mathcal{Y}$ as formal finite sums of Young diagrams with rational coefficients, i.e.,

$$\mathbb{Q}\mathcal{Y} := \left\{ \sum_{i=1}^{k} a_i \lambda^{(i)} :\ k \in \mathbb{N},\ a_i \in \mathbb{Q},\ \lambda^{(i)} \in \mathcal{Y} \right\}.$$

Define two linear "differential" operators on $\mathbb{Q}\mathcal{Y}$. For a Young diagram $\lambda \in \mathcal{Y}$, we have

$$\xi(\lambda) := \sum_{\substack{(i,j)\in\mathbb{N}^2 \\ \lambda'=\lambda\setminus(i,j)\in\mathcal{Y}}} \lambda';$$

and

$$\nabla(\lambda) := \sum_{\substack{(i,j)\in\mathbb{N}^2 \\ \lambda'=\lambda\setminus(i,j)\in\mathcal{Y}}} (j-i)\lambda'.$$

$$\xi\left(\ydiagram{1,3,4}\right) = \ydiagram{1,3,3} + \ydiagram{1,2,4} + \ydiagram{3,4}$$

$$\nabla\left(\ydiagram{1,3,4}\right) = 3\,\ydiagram{1,3,3} + 1\,\ydiagram{1,2,4} - 2\,\ydiagram{3,4}$$

Key Lemma

For the empty diagram, we have $\xi(\emptyset) = \nabla(\emptyset) = 0$, therefore we associate the empty diagram with 1.

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Lemma (N.)

An element from $\mathbb{Q}\mathcal{Y}$ is constant if and only if both operators give zero, i.e.,

$$x \in \mathbb{Q} \iff \xi(x) = \nabla(x) = 0.$$

We say that a map $\star : \mathbb{Q}\mathcal{Y}^2 \to \mathbb{Q}\mathcal{Y}$ is a multiplication if

- $n, m \in \mathbb{N}$ and $x_i \in \mathbb{Q}\mathcal{Y}_i, i \in [0, n], y_j \in \mathbb{Q}\mathcal{Y}_j, j \in [0, m]$,

$$(x_0 + \ldots + x_n) \star (y_0 + \ldots + y_m) = \sum_{\substack{0 \leq i \leq n \\ 0 \leq j \leq m}} x_i \star y_j,$$

  where $x_i \star y_j \in \mathbb{Q}\mathcal{Y}_{(i+j)}$;
- for $a, b \in \mathbb{Q}$, $a \star b = ab$;
- for any $x, y \in \mathbb{Q}\mathcal{Y}$, $\xi(x \star y) = (\xi x) \star y + x \star (\xi y)$;
- for any $x, y \in \mathbb{Q}\mathcal{Y}$, $\nabla(x \star y) = (\nabla x) \star y + x \star (\nabla y)$.

We say that a map $\star : \mathbb{Q}\mathcal{Y}^2 \to \mathbb{Q}\mathcal{Y}$ is a multiplication if

- $n, m \in \mathbb{N}$ and $x_i \in \mathbb{Q}\mathcal{Y}_i, i \in [0, n], y_j \in \mathbb{Q}\mathcal{Y}_j, j \in [0, m]$,

  $$(x_0 + \ldots + x_n) \star (y_0 + \ldots + y_m) = \sum_{\substack{0 \le i \le n \\ 0 \le j \le m}} x_i \star y_j,$$

  where $x_i \star y_j \in \mathbb{Q}\mathcal{Y}_{(i+j)}$;
- for $a, b \in \mathbb{Q}$, $a \star b = ab$;
- for any $x, y \in \mathbb{Q}\mathcal{Y}$, $\xi(x \star y) = (\xi x) \star y + x \star (\xi y)$;
- for any $x, y \in \mathbb{Q}\mathcal{Y}$, $\nabla(x \star y) = (\nabla x) \star y + x \star (\nabla y)$.

Corollary

There is at most one multiplication map.

Theorem (N.)

There is a unique multiplication map.
Furthermore, this map is linear and satisfies commutative and associative properties and it is given by

$$\lambda \star \mu = \sum_{\nu} c^{\nu}_{\lambda,\mu} \nu,$$

where $c^{\nu}_{\lambda,\mu}$ are Littlewood-Richardson coefficients.

Jacobi-Trudi identity

$$h_\ell := \underbrace{\square\square\square\ldots\square\square}_{\ell}$$

Theorem (Jacobi-Trudi identity)

For a partition $\lambda = (\lambda_1 \geq \ldots \geq \lambda_k \geq 0)$, *we have*

$$s_\lambda = \det \begin{bmatrix} h_{\lambda_1} & h_{\lambda_1+1} & h_{\lambda_1+2} & \ldots & h_{\lambda_1+k-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & h_{\lambda_2+1} & \ldots & h_{\lambda_2+k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_k-k+1} & h_{\lambda_k-k+2} & h_{\lambda_k-k+3} & \ldots & h_{\lambda_k} \end{bmatrix}$$

Proof

$$s_\lambda \overset{?}{=} \det_\lambda := \det \begin{bmatrix} h_{\lambda_1} & h_{\lambda_1+1} & h_{\lambda_1+2} & \dots & h_{\lambda_1+k-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & h_{\lambda_2+1} & \dots & h_{\lambda_2+k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_k-k+1} & h_{\lambda_k-k+2} & h_{\lambda_k-k+3} & \dots & h_{\lambda_k} \end{bmatrix}$$

We prove it by induction by $|\lambda| = \lambda_1 + \ldots + \lambda_k$.

Base case: $|\lambda| = 0$. We have $\lambda_1 = \lambda_2 = \ldots = \lambda_k = 0$, therefore $s_\lambda = 1 = det_\lambda$.

Induction step. It is enough to check $\xi(s_\lambda - det_\lambda) = \nabla(s_\lambda - det_\lambda) = 0$.

Proof; Induction step

$$s_\lambda \stackrel{?}{=} \det_\lambda := \det \begin{bmatrix} h_{\lambda_1} & h_{\lambda_1+1} & h_{\lambda_1+2} & \dots & h_{\lambda_1+k-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & h_{\lambda_2+1} & \dots & h_{\lambda_2+k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_k-k+1} & h_{\lambda_k-k+2} & h_{\lambda_k-k+3} & \dots & h_{\lambda_k} \end{bmatrix}$$

We have

$$\xi(h_{\lambda_i-i+j}) = h_{(\lambda_i-1)-i+j},$$

then after combining by rows we get

$$\xi(\det_\lambda) = \sum_{\lambda'=\lambda\setminus(i,\lambda_i)\in\mathcal{Y}} \det_{\lambda'} = \sum_{\lambda'=\lambda\setminus(i,j)\in\mathcal{Y}} s_{\lambda'} = \xi(s_\lambda).$$

Proof; Induction step

$$s_\lambda \stackrel{?}{=} \det_\lambda := \det \begin{bmatrix} h_{\lambda_1} & h_{\lambda_1+1} & h_{\lambda_1+2} & \dots & h_{\lambda_1+k-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & h_{\lambda_2+1} & \dots & h_{\lambda_w+k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_k-k+1} & h_{\lambda_k-k+2} & h_{\lambda_k-k+3} & \dots & h_{\lambda_k} \end{bmatrix}$$

We have

$$\nabla(h_{\lambda_i-i+j}) = (\lambda_i - i + j - 1)h_{\lambda_i-i+j-1} =$$

$$= (\lambda_i - i)h_{(\lambda_i-1)-i+j} + (j-1)h_{\lambda_i-i+(j-1)},$$

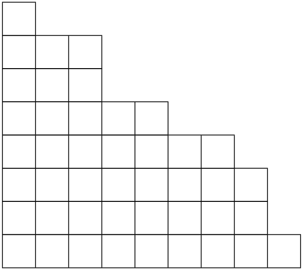
then

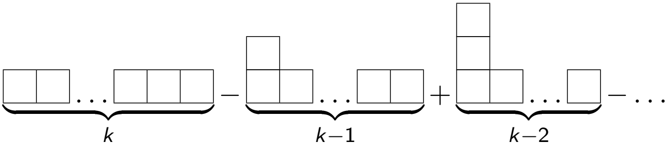
$$\nabla(\det_\lambda) = \sum_{\lambda'=\lambda\setminus(i,\lambda_i)\in\mathcal{Y}} (\lambda_i - i)\det_{\lambda'} = \sum_{\lambda'=\lambda\setminus(i,j)\in\mathcal{Y}} (j-i)s_{\lambda'} = \nabla(s_\lambda).$$

- $\rho^{(1)} := \xi$;
- $\rho^{(k+1)} := \frac{[\rho^{(k)},\nabla]}{k} = \frac{\rho^{(k)}\cdot\nabla - \nabla\cdot\rho^{(k)}}{k}$.

Theorem (N.)

$$\rho^{(k)}\lambda = \sum_{\substack{\mu:\ \mu\subset\lambda,\ |\mu|=|\lambda|-k,\\ \lambda\setminus\mu \text{ is a border strip}}} (-1)^{ht(\lambda\setminus\mu)-1} s_\mu.$$



$p_k :=$ 

$$\rho^{(k)} p_{k'} = k\delta_{k,k'}.$$

Theorem (Murnaghan-Nakayama)

$$s_\lambda p_k = \sum_{\substack{\mu:\ \mu\supset\lambda,\ |\mu|=|\lambda|+k, \\ \mu\backslash\lambda \text{ is a border strip}}} (-1)^{ht(\mu\backslash\lambda)-1} s_\mu.$$

Thank You!