Session 2

Wednesday, March 24

10:45 -10:55	Theo Douvropoulos, University of Massachusetts, Amherst
10:55 - 11:05	Brian Hwang, Cornell University
11:05 - 11:15	Weihong Xu, Rutgers University
11:15 - 11:25	Anastasia Chavez, UC Davis
11:25 - 11:35	Maiko Serizawa, University of Ottawa

Volumes of Root Zonotopes via the W-Laplacian

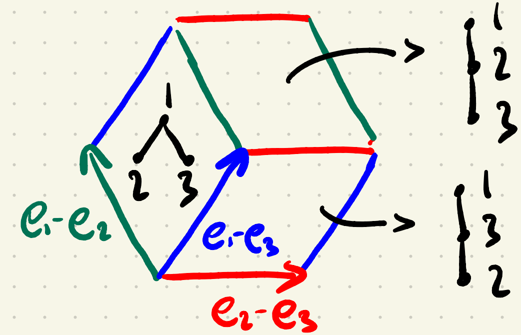
arxiv: 2012.04519

For the "Geometry and Combinatorics from Root Systems" Workshop as part of the Combinatorial Algebraic Geometry semester @ICERM

by Theo Douvropoulos (UMass Amherst)
joint with Guillaume Chapuy

The unimodular root zonotope $Z_{A_{n-1}^+}$ and its volume.

The root zonotope Z_{Φ^+} is the Minkowski sum of the positive roots of the root system Φ.



The root zonotope $Z_{A_{n-1}^+}$ is unimodular and its (normalized) volume is given as $\mathrm{Vol}(Z_{A_{n-1}^+}) = n^{n-2}$

(it has a tiling indexed by trees)

Φ	A_{n-1}	B_2	B_3	B_4	D_3	D_4	F_4	...
$\mathrm{Vol}(Z_{\Phi^+})$	n^{n-2}	7	$3\cdot 29$	1553	2^4	$2\cdot 3\cdot 53$	$2\cdot 3\cdot 31\cdot 67$	...

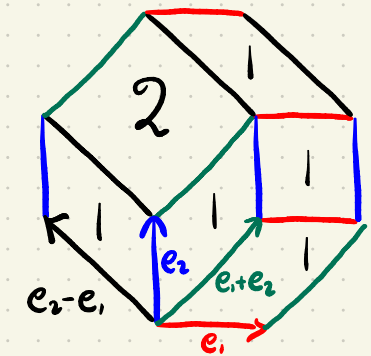
? ? ? ? ? ? ? ? ?

arxiv: 2012.04519

Why not a product formula for all Weyl groups?

arxiv: 2012.04519

- The zonotopes Z_{Φ^+} are not unimodular
- Their tiles are less nice than trees.



$$\mathrm{Vol}(Z_{B_2^+}) = 7$$

[Shephard-McMullen Formula]

$$\mathrm{Vol}(Z_{\Phi^+}) = \sum_{\vec{r}} \overbrace{|\det(\vec{r})|}^{\text{volume of parallelepiped formed by } \vec{r}}$$

$\vec{r}$ ⟶ summing over all $\mathbb{Q}$-bases of Z_{Φ^+}

[Baumeister-Wegener] The reflections associated to $\vec{r}$ generate the Weyl group W if and only if $\vec{r}$ and $\hat{\vec{r}}$ are $\mathbb{Z}$-bases of Q and $\hat{Q}$

Q: root lattice, $\hat{Q}$: coroot lattice

[Corollary]

$$\mathrm{Vol}(Z_{\Phi^+}) = \sum_{W' \leq W} \underbrace{|RGS(W')|}_{\#\text{ of reflection generating sets of } W'} \cdot \underbrace{\mathrm{Vol}(Q')}_{\text{volume of root lattice } Q' \text{ of } W'}$$

W' a max-rank reflection subgroup

The W-Laplacian and its determinant

arxiv: 2012.04519

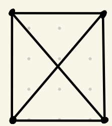
W is a Weyl group acting on $V \simeq \mathbb{R}^n$. It has set of reflections $\mathcal{R}$, root system Φ, and reflection representation ρ_V. Its W-Laplacian L_W is:

Defn 1: $GL(V) \ni L_W := \sum_{\tau \in \mathcal{R}} \left(I_n - \rho_V(\tau) \right)$

↳ the $n \times n$ identity matrix

Defn 2: $L_W(v) := \sum_{\beta \in \Phi^+} \langle v, \hat{\beta} \rangle \cdot \beta$

TYPE-A PICTURE



K_4 graph ⇝ Laplacian

$$\underbrace{\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}}_{L(K_4)} = \sum_{i<j} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \sum_{i<j} \left(I_4 - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)$$

by Defn 1 and W-invariance: $\det(L_W) = h^n$ (h is the Coxeter # of W)

by Defn 2 and as a sum of rank-1 operators:

$$\det(L_W) = \sum_{\vec{r}} \det(\langle r_i, \hat{r}_j \rangle) = \sum_{W' \leq_{max} W} |RGS(W')| \cdot I(W')$$

↳ connection index

Volumes $\mathrm{Vol}(Z_{\Phi^+})$ in terms of Coxeter #s and reflection subgroups

[after Shephard-McMullen] $$\mathrm{Vol}(Z_{\Phi^+}) = \sum_{W' \leq_{max} W} \underbrace{|RGS(W')|}_{\#\text{of reflection generating sets of } W} \cdot \mathrm{Vol}(Q')$$

[after the double calculation of the W-Laplacian] $$|RGS(W')| = \frac{1}{I(W')} \cdot \sum_{W'' \leq_{max} W'} \mu(W', W'') \cdot \underbrace{\prod_{i=1}^{n} h_i(W'')}_{\text{Coxeter \#s of } W''}$$

Möbius function for lattice of max-rank reflection subgroups

Corollary:

$$\mathrm{Vol}(Z_{\Phi^+}) = \sum_{W'' \leq_{max} W} \left(\prod_{i=1}^{n} h_i(W'') \right) \cdot \sum_{W'' \leq W' \leq W} \mu(W', W'') \cdot \frac{1}{\mathrm{Vol}(\hat{Q}')}$$

Corollary: The interval $[W'', W]$ is replaced by $[W'', \text{parab. closure}(W'')]$
(Ehrhart polynomials)

arxiv: 2012.04519

Summary / Advertisement

arxiv: 2012.04519

Existing approaches

- Postnikov $\mathrm{Vol}(Z_{\Phi^+}) = \sum_{w \in W} \cdots$ ⇝ all group elements
- De Concini - Procesi $\mathrm{Vol}(Z_{\Phi^+}) = \sum_{f \in \mathcal{F}} \cdots$ ⇝ $n!$ flags on the simplex.

✱ The above generalize to arbitrary permutohedra.

- Ardila-Beck-McWhirter
  - combinatorics of trees
  - generating functions for Ehrhart polynomials
  - only classical types (A_n, B_n, C_n, D_n)

Our approach

- Debatably more explicit

$$\mathrm{Vol}(Z_{\Phi^+}) = \sum_{W' \leq_{\max} W} \cdots$$

- Reveals connection with a general version of W-trees (i.e. reflection gen'g sets)
- The W-Laplacian plays a "higher-arithmetic" role:

$$\det(L_W) = \sum_{\vec{r}} \left|\det(\vec{r}) \cdot \det(\bar{\vec{r}})\right|$$

Effective point-counting for polygons in flag varieties

Brian Hwang

ICERM Workshop: Geometry and Combinatorics from Root Systems

March 24, 2021

Slides available at: http://brianhwang.com/slides/2021_icerm.pdf

An innocent observation...

Obs 1. A lot of the "big" (e.g. affine, open) spaces that arise when studying flag varieties seem very similar.

Some of these are known to have special features. Two extremal cases:

- Open Richardson varieties $R_u^v = X_u \cap X^v$ (where X_u and X^v are Schubert and opposite Schubert varieties) have an explicit combinatorial decomposition into a *disjoint* union of *finitely many* components of the form $(\mathbf{G}_m)^a \times (\mathbf{A}^1)^b$ for *varying* (a, b).
- Double Bruhat cells $G^{u,v} = B_+\overline{u}B_+ \cap B_-\overline{v}B_-$ are cluster varieties and so admit a combinatorial decomposition into a *highly nondisjoint* union of *possibly infinitely many* components of the form $(\mathbf{G}_m)^d$ for a *fixed* d.

These correspond to two ways in which we can "doubly slice up" a group:

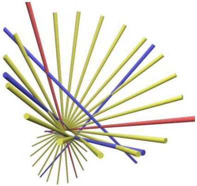
$$G = \bigsqcup_{u \in W} \bigsqcup_{v \in W} R_u^v = \bigsqcup_{u \in W} \bigsqcup_{v \in W} G^{u,v},$$

but are quite different: e.g. $R^{u,v} \neq \varnothing$ if and only if $u \leq v$ but $G^{u,v}$ is always nonempty.

Obs 2. Many of these spaces, including the examples above, are *polygons in flag varieties*, T-fibrations over these, or unions/intersections thereof.

A 'Quantum Equals Classical' Theorem for n-pointed (K-theoretic) Gromov-Witten Invariants of Lines in Homogeneous Spaces

Weihong Xu
(Joint with A. Buch, L. Chen, A. Gibney, L. Heller,
E. Kalashnikov, and H. Larson)



March 2021

Set-up

- Fix $T \subseteq B \subseteq P \subseteq G$, $X = G/P$ flag variety
- $0 \leq \beta \in H_2(X)$
- $ev_i : \overline{M}_{0,n}(X, \beta) \to X$
- Schubert varieties $\Gamma_1, \cdots, \Gamma_n \subseteq X$

  $\mathcal{O}_1, \cdots, \mathcal{O}_n \in K^T(X)$ corresponding T-equivariant Schubert classes
- T-equivariant K-theoretic Gromov-Witten invariant of X

  $I_\beta^T(\mathcal{O}_1, \cdots, \mathcal{O}_n) := \chi^T(ev_1^* \mathcal{O}_1 \cdots ev_n^* \mathcal{O}_n) \in K^T(pt)$

Background

- 3-pointed 'Quantum=Classical' for
  - *Cohomological GW-invariants:*
    - **Buch, Kresch, Tamvakis, '03, '09:** (isotropic) Grassmannians
    - **Chaput, Manivel, Perrin, '08:** cominuscule + type-uniform
    - **Leung, Li, '11:** some invariants for G/P
  - *(Equivariant) K-theoretic GW-invariants:*
    - **Buch, Mihalcea, '11; Chaput, Perrin, '11:** cominuscule G/P
    - **Mihalcea, Li, '13:** most G/P, β class of a line/Schubert curve
- Line case is particularly nice:

  For most choices[1] of G,P, and $\beta = [X_{s_\alpha}]$ $(\alpha \in \Delta \setminus \Delta_P)$

$$\begin{array}{ccc} \overline{M}_{0,1}(X,\beta) & \xrightarrow{ev} & X \\ \downarrow & & \\ \overline{M}_{0,0}(X,\beta) & & \end{array} \quad = \quad \begin{array}{ccc} G/(P\cap Q) & \xrightarrow{p} & G/P \\ {\scriptstyle q}\downarrow & & \\ G/Q & & \end{array}$$

[1]We require that either α is long or the connected component containing α of $\Delta_P \cup \{\alpha\}$ in the Dynkin diagram of G is simply laced. This condition ensures that **lines in the projective embedding of X given by L_{ω_α} are exactly translates of the Schubert curve X_{s_α}.**

n-pointed 'Quantum = Classical' for Lines

Theorem [Buch, Chen, Gibney, X] For most choices[2] of G,P, and β the class of a Schubert curve, the (T-equivariant, n-pointed, genus 0) KGW invariant

$$I_\beta^T(\mathcal{O}_1, \cdots, \mathcal{O}_n) = \chi^T_{G/Q}(q_* p^* \mathcal{O}_1 \cdots q_* p^* \mathcal{O}_n).$$

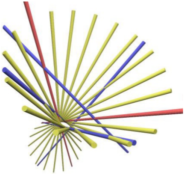
Corollary [direct proof by **C, G, H, K, L, X]** The GW Invariant

$$\int_{\overline{M}_{0,n}(X,\beta)} ev_1^*[\Gamma_1] \cdots ev_n^*[\Gamma_n]$$

$= \int_{G/Q} [q(p^{-1}(\Gamma_1))] \cdots [q(p^{-1}(\Gamma_n))]$
$= \#\{\beta\text{-line in } X \text{ meeting } g_1\Gamma_1, \cdots, g_n\Gamma_n\}$ for $g_1, \cdots, g_n \in G$ general

[2] The same condition as on the previous page.

A Good Old Example



$$\begin{array}{ccc} \begin{tikzcd} Fl(1,2;4) \arrow[r, "p"] \arrow[d, "q"'] & \mathbb{P}^3 \\ Gr(2,4) & \end{tikzcd} & = & \begin{tikzcd} \overline{M}_{0,1}(\mathbb{P}^3,1) \arrow[r, "ev"] \arrow[d] & \mathbb{P}^3 \\ \overline{M}_{0,0}(\mathbb{P}^3,1) & \end{tikzcd} \end{array}$$

Line $\Gamma \subset \mathbb{P}^3$

$q(p^{-1}(\Gamma))$ Schubert divisor $\square$

$\square^4 = \textcolor{red}{2} \cdot \boxplus = \textcolor{red}{2} \cdot [pt]$

$4 \cdot codim(\Gamma) = dim(\overline{M}_{0,4}(\mathbb{P}^3, 1))$

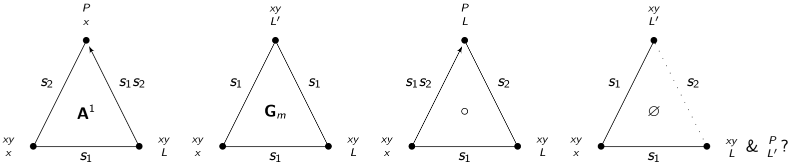
The Gromov-Witten invariant

$\int_{\overline{M}_{0,4}(\mathbb{P}^3,1)} ev_1^*[\Gamma] \cdots ev_4^*[\Gamma] = \textcolor{red}{2}$

Polygons in flag varieties?

Def. A **polygon in a flag variety** is a tuple of flags $(B_1, \ldots, B_m) \in (G/B_+)^m$ together with a tuple of elements $(w_1, \ldots, w_m) \in W$ representing "distances" between adjacent flags: $B_1 \overset{w_1}{\to} B_2, B_2 \overset{w_2}{\to} B_3, \ldots, B_m \overset{w_m}{\to} B_1$.

e.g. For $G = SL_3, PGL_3, GL_3$; $W \cong \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^3 = e \rangle \cong S_3$



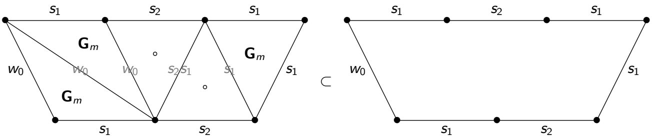
Why nice? These "distances" w_i are sometimes called the *relative position* $B_i \overset{w_i}{\to} B_{i+1}$ of two flags. The distances on G/B_+ and $B_-\backslash G$ together with a "codistance" between them form a *twin building*; it says that a flag variety can essentially be viewed as a compact sphere.

Spaces like double Bruhat cells are *torus fibrations* over polygons in flag varieties, which we parameterize by choosing the addition datum of "compatible" sections of $G/B_+ \to G/U_+$ for each flag B_i.

Key Feature of Polygons in Flag Varieties

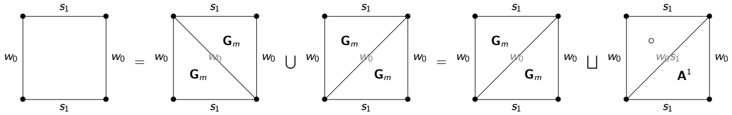
These polygons admit triangulations into *easy-exit* triangles, that is, 3-sided polygons on flag varieties where at least one edge is a simple reflection. Such triangles are isomorphic to $\mathbf{A}^1$, $\mathbf{G}_m$, $\{*\}$, or $\varnothing$, depending on the labels on the other sides and yield iterated $\mathbf{A}^1$- or $\mathbf{G}_m$-fibrations.

e.g. Type A_2, distances $(s_1, s_2, s_1, s_1, s_2, s_1, w_0)$



Upshot. Some of these "easy-exit" triangulations recover decompositions of these spaces.

e.g. Type A, distances (s_i, w_0, s_i, w_0)

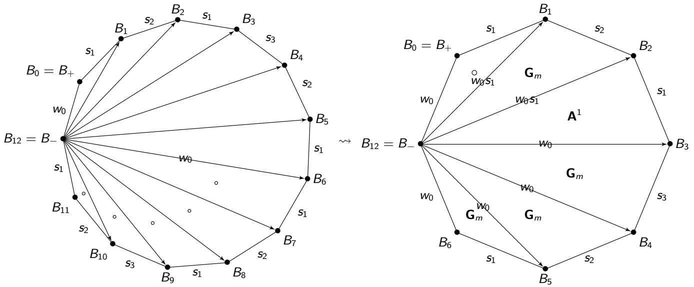


This space is $\mathbf{A}^2 \backslash \{xy - 1\}$. It has $(q-1)^2 + q$ points over $\mathbf{F}_q$.

Application: Effective Point Counting for $R_e^{w_0}$ for $G = SL_4$

Step 1. Set $Q = \mathbf{w} + \mathbf{v}$ where $\mathbf{w}$ and $\mathbf{v}$ are arbitrary reduced expressions for w_0, e.g. $Q = (s_1, s_2, s_1, s_3, s_2, s_1; s_1, s_2, s_1, s_3, s_2, s_1)$

Step 2. Fix a triangulation of the $(|Q| + 1)$-gon and run over the possible W-valued distances on the internal edges:



Rem. The valid distances on the internal diagonals for this "shell" triangulation correspond to *distinguished subexpressions for e* in $\mathbf{w}$ (in sense of Deodhar):

$$121321, \underline{1}2\underline{1}321, 12\underline{1}32\underline{1}, 1\underline{2}13\underline{2}1, \underline{12}13\underline{21}.$$

Only these yield an easy-exit triangulation without a $\varnothing$-triangle.

Step 3. Count points on each component:

$$\#\mathring{\text{Brick}}^Q(\mathbf{F}_q) = (q-1)^6 + 3[q(q-1)^4] + q^2(q-1)^2$$

http://brianhwang.com/slides/2021_icerm.pdf

The main result and looking ahead

Thm. (H.) An effective, embarrassingly parallel, deterministic point-counting algorithm—of complexity $O(n \log n)$ where n is the sum of (the minimum of lengths/co-lengths) of the sides—for polygons in flag varieties over finite fields, where G is a minimal Kac–Moody group.

BIG PICTURE: What properties should a polygon in a flag variety have? At the very least:

1. A "Deodhar-style" decomposition into a disjoint union of $(\mathbf{G}_m)^a \times (\mathbf{A}^1)^b$ components, classified by *distinguished* subexpressions. (Białnicki–Birula decomposition $+\ \epsilon$)

2. A natural ("almost sncd") compactification and a set of *canonical coordinates* that yield a group-theoretic parameterization and moduli-theoretic interpretation of these spaces. (Log Calabi–Yau property $+$ monodromy pairing; alternatively, canonical bases)

Prediction: $\mathbf{1{+}2 = 3}$. *(Conj.)* The coordinate ring of a polygon U in a flag variety is a cluster algebra. Furthermore, we expect that $(U, \overline{U} \backslash U)$ is a log Calabi–Yau pair *with maximal boundary*, with a natural dual with respect to the group $G^\vee$.

Thm. (H.–Knutson) Open brick manifolds—polygons where all but one of the "distances" are *simple reflections*, and G is simply-connected and semisimple—are cluster varieties. In particular, open Richardson varieties for such G.

Matroids, Positroids, and Combinatorial Characterizations

Anastasia Chavez

- Native Californian, living in Berkeley with my partner, two kids and several quirky pets

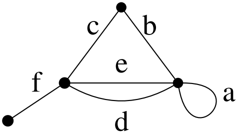
- Finishing as an NSF Postdoc and Krener Assistant Professor, UC Davis, mentored by Dr. Jesus De Loera
- Beginning a tenure-track position at Saint Mary's College of California this summer
- Recently enjoying learning to play the guitar, taking improv classes, and identifying the birds at our bird feeder

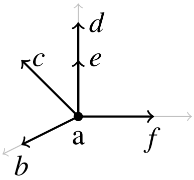
Matroids

Definition (Whitney 1935)

A *matroid* is a pair $M = ([n], \mathcal{B})$ where $B \in \mathcal{B}$ is a basis and $B \subset [n]$ such that

- $\mathcal{B} \neq \emptyset$,
- For all $A, B \in \mathcal{B}$, $a \in A \backslash B \Rightarrow b \in B \backslash A$ s.t. $A - a + b \in \mathcal{B}$.





a	b	c	d	e	f
0	1	1	0	0	0
0	0	1	1	2	0
0	0	0	0	0	1

trees: bases: max. lin. ind. cols:

$$\mathcal{B} = \{bcf, bdf, bef, cdf, cef\}$$

Positroids

- Matroids appear naturally in linear algebra: each $k \times n$ full rank matrix over a field $\mathbf{F}$ gives rise to a matroid on $[n]$ of rank k. Matroids arising in this way are called $\mathbf{F}$-*representable*.

Definition (Postnikov 2005)

A rank k matroid M on $[n]$ is a *positroid* if it is $\mathbb{R}$-representable associated with a totally non-negative full-rank real-$(k \times n)$ matrix.

Example

The matroid $M = ([5], \mathcal{B})$ where $\mathcal{B} = \{13, 14, 15, 34, 35, 45\}$ is a positroid:

$$A_M = \begin{pmatrix} 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}.$$

The matroid $M = ([4], \{12, 14, 23, 34\})$ is not. (why?)

Combinatorics of positroids

Theorem (Postnikov 2006)

A positroid can be represented by a unique: Grassmann necklace, decorated permutation, Le-diagram, and plabic graph.

Define i - *Gale order:* For $S = \{s_1 <_i \cdots <_i s_k\}$,
$T = \{t_1 <_i \cdots <_i t_k\}$ subsets of $[n]$, $S <_i T$ iff $s_j <_i t_j$ for all $j \in [d]$.

Example

- $M(A)$: $\mathcal{B} = \{13, 14, 15, 34, 35, 45\}$
- Grassmann necklace (*list of i-Gale-least bases*):

$$\mathcal{I}_M : (13, 34, 34, 45, 51)$$

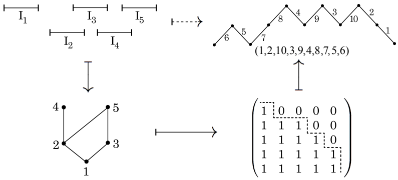
- Decorated permutation (*encodes swaps between necklace elements*):

$$\sigma_M : 4\underline{2}513$$

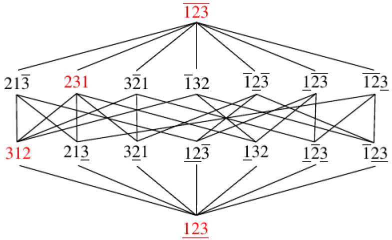
Combinatorial Characterizations

Unit interval positroids: Positroids arising from unit interval orders (a poset) in relation with Dyck paths on associated matrices.

Flag positroids and poset of positroid quotients: Characterization of positroid quotients by applying the *Freeze-Shift-Decorate function* on decorated permutations.



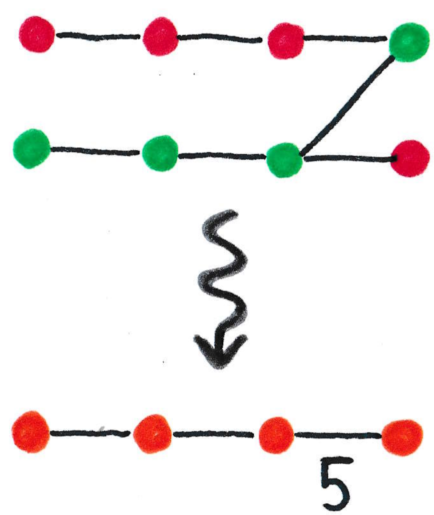
Joint work with Felix Gotti



Joint work with Carolina Benedetti and Daniel Tamayo

Twisted Quadratic Foldings of Root Systems and Combinatorial Schubert Calculus

by Maiko Serizawa



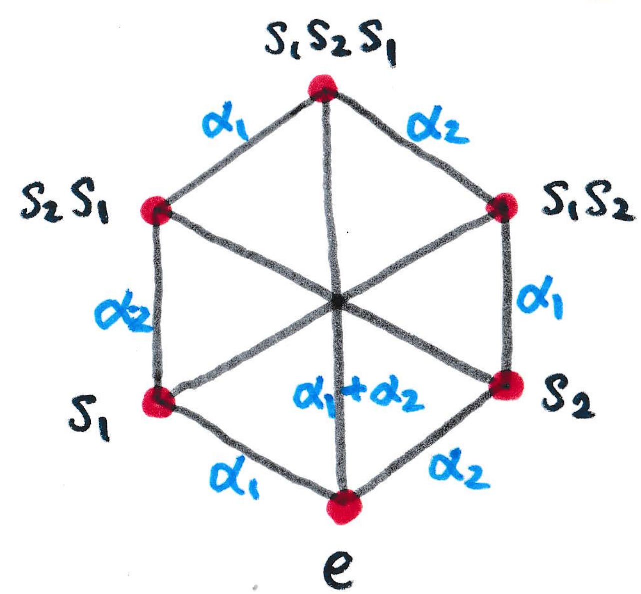
Moment Graph $\mathcal{G}$

W : a finite Coxeter group ($\Leftrightarrow$ a finite real reflection group)

Idea A directed labelled graph on $(W, \leq)$

Bruhat order

e.g. $W = \mathfrak{S}_3 = \langle \underbrace{(1\ 2)}_{= s_1}, \underbrace{(2\ 3)}_{= s_2} \rangle$



α_i ... simple root corresponding to s_i

Structure Algebra $Z(\mathcal{G})$

Idea A certain module structure over $\mathcal{G}$

moment graph

$$Z(\mathcal{G}) \subset \bigoplus_{w \in W} S$$

Cut out by relations associated with the edges of $\mathcal{G}$

$$S = \mathbb{R}[\alpha_1, \cdots, \alpha_n]$$

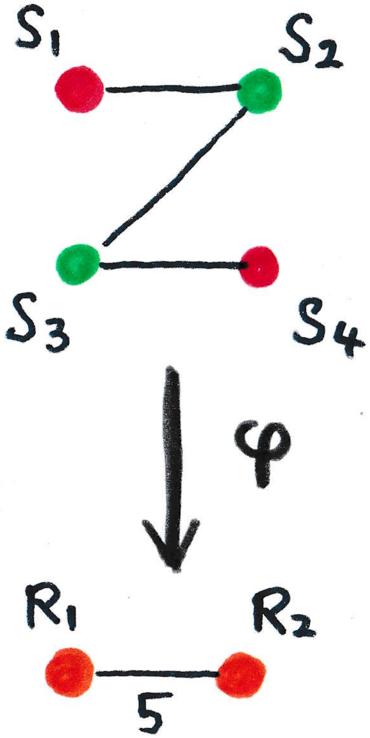
Fact ① $Z(\mathcal{G})$ is a FREE S-module!

$\rightsquigarrow$ Schubert classes

② $\operatorname{rank} Z(\mathcal{G}) = |W|$

③ $Z(\mathcal{G})$ is an S-algebra under the pointwise multiplication.

Folding and Induced Map



$W = \mathfrak{S}_5$

$W_\tau = D_{10}$

Embedding of Coxter groups

$$W_\tau \xhookrightarrow{\varepsilon} W$$

Induced map

$$\begin{array}{ccc} Z(\mathfrak{g}) & \xrightarrow{\varepsilon^*} & Z(\mathfrak{g}_\tau) \\ \cup & & \cup \\ \{\sigma^{(w)} \mid w \in W\} & & \{\sigma_\tau^{(u)} \mid u \in W_\tau\} \end{array}$$

Schubert classes

Q. Let $u \in W_\tau$. ($\rightsquigarrow \sigma_\tau^{(u)}$)

Is there $w \in W$ such that $\varepsilon^*(\sigma^{(w)}) = c \cdot \sigma_\tau^{(u)}$ for some nonzero $c \in \mathbb{R}$?

Main Result

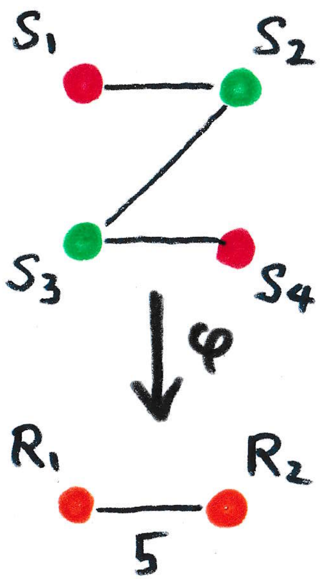
Theorem (S. 2020)

Let $u \in W_\tau$ and $w \in W$. Then

$$\varepsilon^*(\sigma^{(w)}) = c \cdot \sigma_\tau^{(u)} \quad \text{for some } c \in \mathbb{R}^\times$$

$$\iff \quad w \in W^F \quad \text{and} \quad u = \overline{\varphi}(w)$$

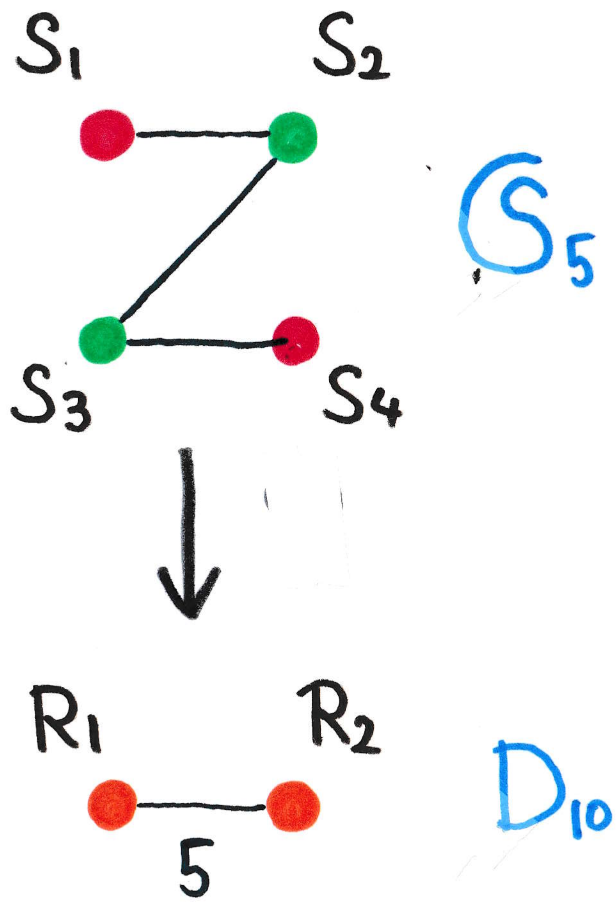
e.g.



$$\begin{aligned} \overline{\varphi} : W^F &\longrightarrow W_\tau \\ e &\longmapsto e \\ w = s_{i_1} \cdots s_{i_\ell} &\longmapsto \varphi(s_{i_1}) \cdots \varphi(s_{i_\ell}) \end{aligned}$$

any reduced expression!

Example



	$u \in D_{10}$	$w \in (\mathfrak{S}_5)^F$ & $\bar{\varphi}(w) = u$
0	e	e
1	R_1 R_2	$s_3, \quad s_1$ $s_4, \quad s_2$
2	R_2R_1 R_1R_2	$s_2s_1, \; s_2s_3, \; s_4s_3$ $s_1s_2, \; s_3s_2, \; s_3s_4$
3	$R_1R_2R_1$ $R_2R_1R_2$	$s_1s_2s_3, \quad s_3s_2s_1$ $s_2s_3s_4, \quad s_4s_3s_2$
4	$R_2R_1R_2R_1$ $R_1R_2R_1R_2$	$s_4s_3s_2s_1$ $s_1s_2s_3s_4$
5	$R_1R_2R_1R_2R_1$	—

By Theorem, for these u and w, we have $\mathcal{E}^*(\sigma^{(w)}) = c \cdot \sigma_\tau^{(u)}$