Doing Schubert Calculus with Bumpless Pipe Dreams

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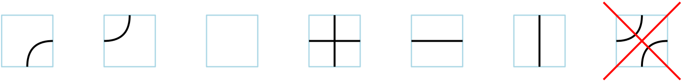
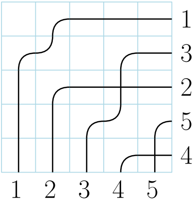
Basics of bumpless pipe dreams

Definition (Lam-Lee-Shimozono '18). An S_n-**bumpless pipe dream** is a tiling of an $n \times n$ grid by the following six kinds of tiles



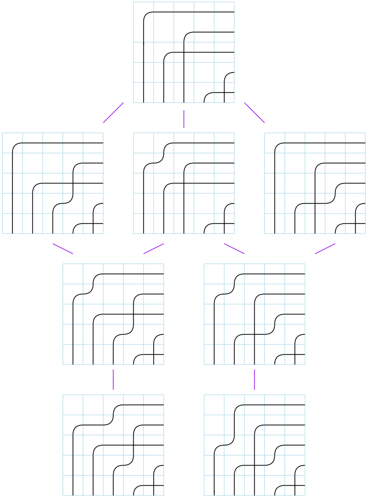
such that n pipes travel from the south edge and exit from the east edge, and no two pipes cross twice.

Example. A bumpless pipe dream for $13254 \in S_5$



Basics of bumpless pipe dreams

- The set of bumpless pipe dreams for a permutation π is denoted as BPD(π).
- The **Rothe bumpless pipe dream** of π is the bumpless pipe dream of π that looks like the Rothe diagram of π.
- All bumpless pipe dreams of π can be obtained from the Rothe bumpless pipe dream by performing a sequence of **droop moves**.
- In a bumpless pipe dream, there are always as many blank tiles as there are crosses.



BPDs compute (double) Schubert polynomials

Definition-Theorem (Lam-Lee-Shimozono '18)

The *bumpless pipe dream polynomial* for $\pi \in S_\infty := \bigcup_n S_n$ is defined as

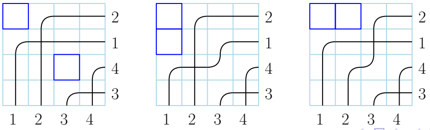
$$\mathfrak{S}_\pi(\mathbf{x}, -\mathbf{y}) := \sum_{D \in \mathrm{BPD}(\pi)} \prod_{(i,j) \in \mathrm{blank}(D)} (x_i - y_j).$$

It is the same as the double Schubert polynomial for π. Setting $\mathbf{y} = 0$, we get Schubert polynomials.

Example:

$$\mathfrak{S}_{2143}(\mathbf{x}, -\mathbf{y}) = (x_1 - y_1)(x_3 - y_3) + (x_1 - y_1)(x_2 - y_1) + (x_1 - y_1)(x_1 - y_2)$$

$$\mathfrak{S}_{2143}(\mathbf{x}) = x_1 x_3 + x_1 x_2 + x_1^2$$



Basics of Schubert calculus

- The classes of Schubert varieties $\{[X_\pi]\}_{\pi\in S_n}$ form a $\mathbb{Z}$-linear basis of

$$H^*(Fl(\mathbb{C}^n),\mathbb{Z}) \cong \mathbb{Z}[x_1,\cdots,x_n]/I,$$

  where $I = \langle\text{symmetric functions with no constant terms}\rangle$.
- Schubert polynomials $\mathfrak{S}_\pi$ are nice representatives of Schubert classes.
- $\mathfrak{S}_\pi(\mathbf{x})\mathfrak{S}_\rho(\mathbf{x}) = \sum_\sigma c^\sigma_{\pi,\rho}\mathfrak{S}_\sigma(\mathbf{x})$. The **Schubert structure constants** $c^\sigma_{\pi,\rho}$ are nonnegative integers.

In the T-equivariant setting,

- $H^*_T(Fl(\mathbb{C}^n),\mathbb{Z})$ is a free module over $H^*_T(pt,\mathbb{Z}) = \mathbb{Z}[y_1,\cdots,y_n]$ generated by the T-equivariant Schubert classes $\{[X_\pi]_T\}_{\pi\in S_n}$
- Double Schubert polynomials represent T-equivariant Schubert classes.

Monk's rule for Schubert polynomials

Theorem (Monk, 1959)

Let $\pi \in S_\infty$, $\alpha \geq 1$.

$$\begin{aligned}\mathfrak{S}_\alpha(\mathbf{x})\mathfrak{S}_\pi(\mathbf{x}) &= (x_1 + x_2 + \cdots + x_\alpha)\mathfrak{S}_\pi(\mathbf{x}) \\ &= \sum_{\substack{k \leq \alpha < \ell \\ \pi t_{k,\ell} \gtrdot \pi}} \mathfrak{S}_{\pi t_{k,\ell}}(\mathbf{x})\end{aligned}$$

Subtracting $\mathfrak{S}_{\alpha-1}(\mathbf{x})\mathfrak{S}_\pi(\mathbf{x})$ and rearranging, we get

$$x_\alpha \mathfrak{S}_\pi(\mathbf{x}) + \sum_{\substack{k < \alpha \\ \pi t_{k,\alpha} \gtrdot \pi}} \mathfrak{S}_{\pi t_{k,\alpha}}(\mathbf{x}) = \sum_{\substack{\alpha < \ell \\ \pi t_{\alpha,\ell} \gtrdot \pi}} \mathfrak{S}_{\pi t_{\alpha,\ell}}(\mathbf{x})$$

- When the sum on the r.h.s has only one summand, we get the *transition formula*.
- When the sum on the l.h.s is empty, we get the *co-transition formula*.

Monk's rule expressed with bumpless pipe dreams

Theorem (H. '20)

Given $\pi \in S_n$ and $1 \leq \alpha < n$ such that there exists $\ell > \alpha$ where $\pi t_{\alpha,\ell} \gtrdot \pi$, there exists a bijection

$$\Phi_\pi : \{x_\alpha\} \times \mathrm{BPD}(\pi) \sqcup \coprod_{\substack{k<\alpha \\ \pi t_{k,\alpha} \gtrdot \pi}} \mathrm{BPD}(\pi t_{k,\alpha}) \to \coprod_{\substack{\alpha<\ell \\ \pi t_{\alpha,\ell} \gtrdot \pi}} \mathrm{BPD}(\pi t_{\alpha,\ell})$$

that proves Monk's rule bijectively.

Context and Implications

- Weigandt '20 showed an "easy" bijection of the equivariant transition rule with bumpless pipe dreams
- Knutson '19 showed an "easy" bijection of the equivariant co-transition rule with ordinary pipe dreams
- Billey-Holroyd-Young '12 gave a "hard" non-equivariant bijective proof of the transition rule with ordinary pipe dreams
- With either the (non-equivariant) transition or co-transition bijections on pipe dreams and bumpless pipe dreams, one can construct inductively bijections between BPD(π) and PD(π)

Monk's rule for double Schubert polynomials

Theorem (Monk's rule for double Schubert polynomials)

Let $\pi \in S_\infty$, $\alpha \geq 1$.

$$\mathfrak{S}_\alpha(\mathbf{x}, -\mathbf{y})\mathfrak{S}_\pi(\mathbf{x}, -\mathbf{y}) = \sum_{\substack{k \leq \alpha < \ell \\ \pi\, t_{k,\ell} \gtrdot \pi}} \mathfrak{S}_{\pi\, t_{k,\ell}}(\mathbf{x}, -\mathbf{y}) + \sum_{i=1}^{\alpha} (y_{\pi(i)} - y_i)\mathfrak{S}_\pi(\mathbf{x}, -\mathbf{y})$$

Subtracting $\mathfrak{S}_{\alpha-1}\mathfrak{S}_\pi$ and rearranging, we get

$$(x_\alpha - y_{\pi(\alpha)})\mathfrak{S}_\pi(\mathbf{x}, -\mathbf{y}) + \sum_{\substack{k < \alpha \\ \pi\, t_{k,\alpha} \gtrdot \pi}} \mathfrak{S}_{\pi\, t_{k,\alpha}}(\mathbf{x}, -\mathbf{y}) = \sum_{\substack{\alpha < \ell \\ \pi\, t_{\alpha,\ell} \gtrdot \pi}} \mathfrak{S}_{\pi\, t_{\alpha,\ell}}(\mathbf{x}, -\mathbf{y})$$

Q: Can this be proved bijectively?

A: Yes! With *decorated bumpless pipe dreams*, where each blank tile is decorated with a binary label, “x” or “−y”.

Monk's rule for double Schubert polynomials

Let $\widetilde{\mathrm{BPD}}(\pi) := \{(D,f) : D \in \mathrm{BPD}(\pi), f : \mathit{blank}(D) \to \{\mathrm{x}, -\mathrm{y}\}\}$ be the set of decorated bumpless pipe dreams for π. Then

$$\mathfrak{S}_\pi(\mathrm{x}, -\mathrm{y}) = \sum_{(D,f)\in\widetilde{\mathrm{BPD}}(\pi)} \prod_{\substack{(i,j)\in \mathit{blank}(D)\\ f(i,j)=\mathrm{x}}} x_i \prod_{\substack{(i,j)\in \mathit{blank}(D)\\ f(i,j)=-\mathrm{y}}} (-y_i)$$

Theorem (H. '20)

Given $\pi \in S_n$ and $1 \le \alpha < n$ such that there exists $\ell > \alpha$ where $\pi t_{\alpha,\ell} \gtrdot \pi$, there exists a bijection

$$\widetilde{\Phi}_\pi : (\{\mathrm{x}, -\mathrm{y}\} \times \widetilde{\mathrm{BPD}}(\pi)) \sqcup \coprod_{\substack{k<\alpha\\ \pi\, t_{k,\alpha} \gtrdot \pi}} \widetilde{\mathrm{BPD}}(\pi\, t_{k,\alpha}) \to \coprod_{\substack{\alpha<\ell\\ \pi\, t_{\alpha,\ell} \gtrdot \pi}} \widetilde{\mathrm{BPD}}(\pi\, t_{\alpha,\ell})$$

that proves Monk's rule for double Schubert polynomials bijectively.

Can bumpless pipe dreams tell us something about Schubert calculus we didn't know before?

The separated descent Schubert problems

Definition

The **descents** of a permutation π is the set

$$\mathrm{Des}(\pi) := \{i : \pi(i) > \pi(i+1)\}$$

Example. $\pi = 5|13|246$, $\mathrm{Des}(\pi) = \{1, 3\}$

Definition (Knutson–Zinn-Justin)

Two permutations π and ρ have **separated descents at position** k if π has no descents before position k and ρ has no descents after position k.

Example. $\pi = 135|26|4, \rho = 5|14|236$, $k = 3$
Non-Example. $\pi = 14|3|2$, $\rho = 2|14|3$

This condition defines a subclass of the Schubert problem $\mathfrak{S}_\pi \mathfrak{S}_\rho = \sum_\sigma c^\sigma_{\pi,\rho} \mathfrak{S}_\sigma$ where π and ρ have separated descents.

The context

- **Grassmannian Schubert problem**: π and ρ both have a single descent at position k (e.g. $k = 3, \pi = 135|246$, $\rho = 236|145$) (Littlewood-Richardson 1937, made correct decades later, many different rules), *generalized by...*
- **Kogan Schubert problem**: π has a single descent at position k, ρ has descents either at or before position k, or at or after position k (e.g. $k = 3, \pi = 135|246$, $\rho = 26|3|145$) (Kogan '00, with different rules later given by Knutson-Yong '04, Lenart '10, Assaf '17), *generalized by...*
- **Separated descent Schubert problem**: defined and solved by Knutson–Zinn-Justin '19 with puzzles using theory of quiver varieties (not yet published). Our rule is a tableaux/BPD rule using elementary methods.

Schubert products for permutations with separated descents

Theorem (H.)

Let $\pi, \rho \in S_n$ where π has no descents before position k and ρ has no descents after position k. Define

$$\pi \star \rho(i) = \begin{cases} \pi(i+k) - k & \text{if } i \in [1-k, 0] \\ \rho(i) + n - k & \text{if } i \in [1, k] \\ \pi(i) - k & \text{if } i \in [k+1, n] \\ \substack{(i-n)\text{th smallest number in} \\ [1-k, 2n-k] \setminus \pi\star\rho([1-k,n])} & \text{if } i \in [n+1, 2n-k]. \end{cases}$$

Let $\sigma \in S_{2n-k}$ such that $\ell(\pi \star \rho) - \ell(\sigma) = \ell((\pi \star \rho)\sigma^{-1}) = k(n-k)$.The Schubert structure constant $c^{\sigma}_{\pi,\rho}$ is equal to the number of reduced word tableaux T of shape $k \times (n-k)$ such that the permutation associated to T is $(\pi \star \rho)\sigma^{-1}$. Furthermore, $c^{\sigma}_{\pi,\rho} = 0$ for all other σ.

Example

Let $n = 6$, $k = 3$, $\pi = 135|26|4, \rho = 5|13|246$, then
$\pi \star \rho = [-2, 0, 2, 8, 4, 6, -1, 3, 1, 5, 7, 9]$.

$$\begin{aligned}\mathfrak{S}_\pi \mathfrak{S}_\rho = & \mathfrak{S}_{615243} + \mathfrak{S}_{534162} + \mathfrak{S}_{625134} + \mathfrak{S}_{526143} + 2\mathfrak{S}_{624153} \\ & + \mathfrak{S}_{7152346} + \mathfrak{S}_{7142536} + \mathfrak{S}_{7231546}\end{aligned}$$

For $\sigma = 624153$, there are two Coxeter-Knuth classes of reduced words for $(\pi \star \rho)\sigma^{-1}$ whose reduced word tableaux are of shape 3×3. These are

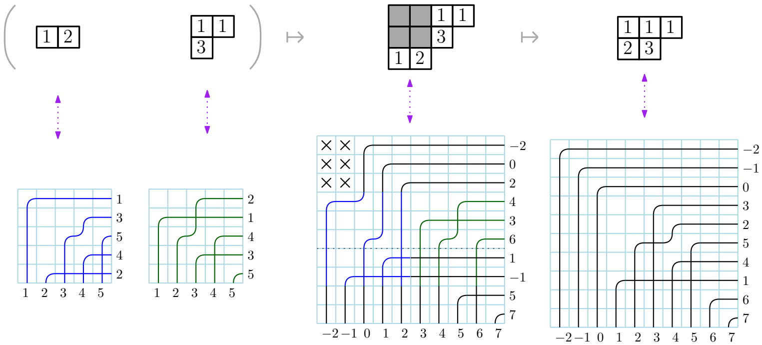
−1	1	3
0	4	5
2	6	7

−1	1	3
0	2	5
4	6	7

For $\sigma = 7142536$, there is one Coxeter-Knuth class of reduced words for $(\pi \star \rho)\sigma^{-1}$ whose reduced word tableau is of shape 3×3:

−1	1	3
0	2	5
1	4	7

The idea



1 2
1 1
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1 1
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1 2
1 1 1
2 3
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5
4
2
1 2 3 4 5
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1
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3
5
1 2 3 4 5
−2
0
2
4
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6
1
−1
5
7
−2 −1 0 1 2 3 4 5 6 7
−2
−1
0
3
2
5
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−2 −1 0 1 2 3 4 5 6 7