Rigid local systems and the multiplicative eigenvalue problem

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Plan

(a) Rigid local systems (definitions and problem of rigid local systems with unitary, and finite global monodromy).

(b) The multiplicative eigenvalue problem, and the problem of vertices of eigenpolytopes (quantum generalizations of Littlewood-Richardson cones).

(c) The relationship between (a), (b) via strange duality.

(d) Consequences for rigid local systems.

(e) Method of proof and computation of vertices.

Rigid local systems

A rank ℓ local system on $\mathbb{P}^1_{\mathbb{C}} - S$, $S = \{p_1, \dots, p_s\}$, is a s-tuple $(A_1, \dots, A_s)$ of matrices $A_i \in \mathrm{GL}(\ell, \mathbb{C})$ with product $A_1 A_2 \cdots A_s = I$. Such tuples are taken up to conjugacy, i.e., $(A_1, \dots, A_s) \sim (CA_1C^{-1}, CA_2C^{-1}, \dots, CA_sC^{-1})$.

They are identified with reps $\rho : \pi_1(\mathbb{P}^1_{\mathbb{C}} - S, b) \to \mathrm{GL}(\ell, \mathbb{C})$, with $\rho(\text{loop around } p_i) = A_i$. The image of ρ is called the (global) monodromy group of the local system.

- The local system is irreducible if ρ is irreducible.
- The local system is rigid, if any other local system with the same local monodromies B_i (i.e, B_i is conjugate to A_i) is conjugate to it. That is there is a single C such that $CB_iC^{-1} = A_i$.

The rigidity condition on an irreducible local system can be captured by a numerical equation ("expected dimension =0").

Questions

This concept has origins in the work of Riemann on hypergeometric functions. Katz shows all (irreducible) rigid local systems are produced by an inductive, algorithmic procedure (middle convolution and tensoring). Some questions:

- Classify unitary rigid local systems (image in $U(\ell)$). How many are there? Assume that all local monodromies are nth roots of unity. In general, all rigids land in a $U(p, q)$, $p + q = \ell$.
- Classify local systems with finite global monodromy (has classical origins in the work of Schwarz). Obviously this is a subset of the previous case (and is determined from it by Galois conjugation).

Note that all local monodromies are *nth* roots of identity in the rigid local system problem.

A rank 2 hypergeometric example from Schwarz's list

$$z(1-z)w'' + (\frac{1}{2} - z)w' + \frac{1}{4}w = 0$$

which corresponds to the matrix equation ($i = \sqrt{-1}$)

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = I.$$

The global monodromy is a finite group (dihedral). It is hence also unitary. All three matrices are conjugate, with eigenvalues i and $-i$.

The multiplicative eigenvalue problem

Conjugacy classes in $\mathrm{SU}(n)$ are in $1-1$ correspondence with points of the simplex
$\Delta_n = \{a = (a_1, \ldots, a_n) \mid a_1 \geq \cdots \geq a_n \geq a_1 - 1, \sum_{j=1}^n a_j = 0\}$
(The correspondence takes a to the conjugacy class of the diagonal matrix with entries $\exp(2\pi\sqrt{-1}a_j)$.

Define $P_n(s) \subset \Delta_n^s$ to be the set of tuples $\vec{a} = (a^1, a^2, \ldots, a^s)$ such that there exist $A_1, A_2, \ldots, A_s \in \mathrm{SU}(n)$ with A_i in the conjugacy class corresponding to a^i, with $A_1 A_2 \cdots A_s = I_n \in \mathrm{SU}(n)$.

Vertices

- $P_n(s)$ parameterizes all tuples of local monodromies of rank n special unitary local systems on $\mathbb{P}^1 - S$.
- $P_n(s)$ is a compact polytope cut out by a finite set of inequalities controlled by quantum Schubert calculus of Grassmannians.
- $P_n(s)$ is the quantum generalization of the Littlewood Richardson cone controlling summands in the tensor products of representations of $\mathrm{SL}(n)$.

Question: What are the vertices of $P_n(s)$? (They all correspond to reducible local systems). Do they have any geometric structure, and special significance? (Thaddeus, Agnes 2014 for a list).
Two types of vertices: Ones for which there is an essentially only one solution to the matrix equation ("F-vertices").

Vertices

There are trivial vertices: $\prod(\zeta^{m_j} I) = I$ where $\zeta = \exp(2\pi i/n)$ and n divides $\sum m_i$.

The first example of a nontrivial-vertex: The following is a vertex for $n = 4$ and $|S| = 3$:
$\mathrm{diag}(1, 1, -1, -1)\,\mathrm{diag}(1, -1, 1, -1)\,\mathrm{diag}(1, -1, -1, 1) = I_4$.

Theorem

Rigid (irreducible) local systems (arbitrary rank) which are unitary with local monodromy of order dividing n are in one-one correspondence with F- vertices of $P_n(s)$.

This correspondence takes the rigid local system of rank 2 to the above vertex (of a rank 4 problem).

Description of the correspondence

Let $\mathcal{A} = (A^1, \dots, A^s)$ correspond to a rank ℓ unitary rigid (irreducible) local system (with local monodromy nth roots of unity). Assume that the eigenvalues of A_i are $\exp(2\pi\sqrt{-1}\mu^i_j/n)$ with integers $0 \leq \mu^i_j < n$, so that the $\mu^i = (\mu^i_1, \dots, \mu^i_\ell)$ are Young diagrams that fit in an $\ell \times n$ box.

Consider the vector of transpose Young diagrams $\vec{\lambda} = (\lambda^1, \dots, \lambda^s)$ with $\lambda^i = (\mu^i)^T$. Define points $a^i \in \Delta_n$ as follows (c_i are chosen to make the traces zero) $a^i = \frac{\lambda^i}{\ell} - c_i(1, 1, \dots, 1)$.

The correspondence takes the local system $\mathcal{A}$ to the point $v(\mathcal{A}) = \vec{a} \in P_n(s)$.

An example

The example of the rank 2 rigid had all local monodromies conjugate to $\mathrm{diag}(i, -i)$. Take $n = 4$, and $\mu = (3, 1)$ a Young diagram that fits into a 2×4 box. Now take the transpose, we get $(2, 1, 1, 0)$. Therefore the a's are all equal to $(1/2, 0, 0, -1/2)$ which corresponds to the diagonal matrix $\mathrm{diag}(1, 1, -1, -1)$.

Origin of the correspondence: Grassmann duality, or strange duality

Let $\mathcal{P}\mathrm{ar}_\ell$ be the moduli stack of rank ℓ parabolic bundles on $\mathbb{P}^1$ with parabolic structures at points of S. $\mathcal{A}$ gives a tuple of representations of $GL(\ell)$ at level n, thus a line bundle on $\mathcal{P}\mathrm{ar}_\ell$. There exists an actual unitary local system with this local monodromy data iff $h^0(\mathcal{P}\mathrm{ar}_\ell, \mathcal{L}) \neq 0$ (and quantum saturation, ignoring determinant issues) .

- The transpose of the data gives a line bundle on $\mathcal{P}\mathrm{ar}_n$ at level ℓ which is also effective (has space of sections dual to the one in the previous paragraph).
- Explained by $h^0(\mathcal{P}\mathrm{ar}_\ell, \mathcal{L}) =$ structure constant in cohomology of Grassmannians $\mathrm{Gr}(n, n+\ell)$ and we use $\mathrm{Gr}(n, n+\ell) = \mathrm{Gr}(\ell, n+\ell)$ to get $h^0(\mathcal{P}\mathrm{ar}_\ell, \mathcal{L}) = h^0(\mathcal{P}\mathrm{ar}_n, \mathcal{L}^T)$

All in all, the point $\mathcal{A}$ lies in P_ℓ iff the transposed point lies in $P_n(s)$. This last correspondence (extremely non-linear) takes the vertex property on one side to the rigidity property on the other.

Some consequences

- There are only finitely many unitary (similarly finite) rigid local systems (arbitrary rank) with local monodromies nth roots of unity (since there are only finitely many vertices of $P_n(s)$).
- Any finite monodromy rigid local system with local monodromies nth roots of unity has a special properties: Eigenvalues at any point do not have ratios that are primitive roots of unity. Hence there are no such local systems of rank > 1 if n is prime (answers a question of Nick Katz).

The second property comes from viewing the corresponding line bundle on the $P_n(s)$ side. It is a natural degeneracy locus, and the corresponding cycle class (computed enumeratively) has a special non-adjacency property.

Work on determining vertices

The vertices of $P_n(s)$ have been inductively determined: The set of vertices on a regular face of $P_n(s)$ has the following description (this generalizes earlier work on the classical counterpart: Vertices of the Klyachko cone, and joint work with Josh Kiers for arbitrary groups)

- Some are given geometrically as natural enumerative loci on $\mathcal{P}\mathrm{ar}_n$. (e.g., the locus of parabolic bundles which have a subbundle with overdetermined number of conditions by 1 on how their fibers look like)
- The rest arise from an explicit induction from Levi subgroups.

Questions

1. Generalize from $U(n)$ to $U(p, q)$ with $p \leq m$.
2. Relations with Katz's algorithm (which does not stay inside the unitary world). Is there a way to run Katz's algorithm staying entirely inside unitary local systems?
3. Problems for other types: Strange duality accompanies conformal embeddings of groups. Is there a generalization for rigid G-local systems? Ongoing joint work with Josh Kiers on the multiplicative eigenvalue problem for arbitrary type.