Gröbner Geometry of Schubert Polynomials Through Ice

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Based on joint work with Oliver Pechenik and Zachary Hamaker and work in progress with Patricia Klein.
Schubert Polynomials
The complete flag variety $GL(n)/B$ has a special family of subvarieties $\{X_\omega : \omega \in S_n\}$ called Schubert varieties.

Each Schubert variety defines a Schubert class $\sigma_\omega \in H^*(GL(n)/B)$. The Schubert classes are a linear basis for this ring.

By Borel's isomorphism:

$$H^*(GL(n)/B) \cong \mathbb{Z}[x_1, x_2, ..., x_n]/I^{S_n}.$$
Lascoux and Schützenberger (1982) defined \( \text{Schubert polynomials} \ \{G_w(x) : w \in S_n \} \) which are a choice of coset representatives for the images of the Schubert classes under Borel's isomorphism.

There's also \textit{double Schubert polynomials} \( \{G_w(x; y) : w \in S_n \} \) which are enriched versions of single Schuberts and satisfy \( G_w(x) = G_w(x; 0) \).
Pipe Dreams

On main antidiagonal

Above main antidiagonal

Below main antidiagonal

Fill nxn grid with n pipes that start at the top and end at the left and pairwise cross at most one time.

\[ Wt(P) = (x_1 - y_1)(x_1 - y_2)(x_1 - y_3)(x_2 - y_1)(x_2 - y_4)(x_3 - y_1)(x_3 - y_2) \]
Theorem (Fomin-Kirillov 1996/Bergeron-Billey 1993):

The double Schubert polynomial is

\[ G_w(x; y) = \sum_{P \in \text{Pipes}(w)} wt(P). \]

\[ G_{2143}(x; y) = (x_1 - y_1)(x_3 - y_1) + (x_1 - y_1)(x_2 - y_2) + (x_1 - y_1)(x_1 - y_2) \]
Bumpless Pipe Dreams

Fill an $n \times n$ grid with $n$ pipes that start at the bottom, end at the right, and pairwise cross at most one time.

\[ \text{wt}(P) = (x_1 - y_1)(x_1 - y_2)(x_1 - y_3) \cdot (x_2 - y_1)(x_2 - y_2)(x_3 - y_1)(x_3 - y_5) \]

no bumps allowed!
Theorem (Lam-Lee-Shimozono 2018):

The double Schubert polynomial is

$$G_w(x; y) = \sum_{P \in \text{BPD}(w)} \text{wt}(P).$$

$$G_{2143}(x; y) = (x_1 - y_1)(x_3 - y_3) + (x_1 - y_1)(x_2 - y_1) + (x_1 - y_1)(x_1 - y_2)$$
Secretly, BPDs were hiding in an unpublished preprint of Lascoux (2002). Lascoux gave a formula for double Grothendieck polynomials as a sum over states of the 6-vertex (ice) model.

\[ \Sigma_{2143}(x; y) = (x_1 - y_1)(x_3 - y_3) + (x_1 - y_1)(x_2 - y_2) + (x_1 - y_1)(x_1 - y_2) \]

\[ \Sigma_{2143}(x) = x_1 x_3 + x_1 x_2 + x_1^2 \]

\[ \Sigma_{2143}(x; y) = (x_1 - y_1)(x_3 - y_3) + (x_1 - y_1)(x_2 - y_1) + (x_1 - y_1)(x_1 - y_2) \]
Matrix
Schubert
Varieties
Let $\text{Mat}(m,n)$ be the space of $m \times n$ matrices and

$$Z_{m \times n} = \begin{bmatrix}
Z_{11} & Z_{12} & \cdots & Z_{1n} \\
Z_{21} & Z_{22} & \cdots & Z_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
Z_{m1} & Z_{m2} & \cdots & Z_{mn}
\end{bmatrix}$$

a generic matrix.

$R = \mathbb{C}[Z_{m \times n}] = \mathbb{C}[Z_{11}, Z_{12}, \ldots, Z_{mn}]$ is the coordinate ring of $\text{Mat}(m,n)$. 
Let $I_k(Z_{m,n})$ be the ideal generated by the minors of size $k$ in $Z_{m,n}$. $I_k(Z_{m,n})$ is a determinantal ideal.

Example:

$I_2 \left( \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \end{bmatrix} \right) = \langle |Z_{11} Z_{12}|, |Z_{11} Z_{13}|, |Z_{12} Z_{13}| \rangle$

$V(R/I_k(Z_{m,n}))$ is the set of matrices $M \in \text{Mat}(m,n)$ such that $\text{rank}(M) \leq k$. 
Given \( w \in S_n \), the Schubert determinantal ideal \( I_w \leq R = \mathbb{C}[Z_{n \times n}] \) is generated by minors of varying size, determined by \( w \).

\[
I_{426135} = \langle z_{11}, z_{12}, z_{13}, z_{21}, z_{31} \rangle + \langle \text{2x2 minors in } z_{ij} \rangle + \langle \text{3x3 minors in } z_{ij} \rangle
\]
Matrix Schubert Varieties

\[ \mathfrak{gl}(n) \xrightarrow{\pi} \text{Mat}(n) \]

\[ \pi^{-1}(X_\omega) \xrightarrow{i} \mathcal{C}(\pi^{-1}(X_\omega)) \]

Matrix Schubert Variety

\[ \pi \downarrow \]

\[ \mathfrak{gl}(n)/B \xrightarrow{\subset} \text{Flag Variety} \]

\[ X_\omega \subset \text{Schubert Variety} \]

Write \( \overline{X_\omega} := \mathcal{C}(\pi^{-1}(X_\omega)) \).

Theorem (Fulton 1992): \( \overline{X_\omega} \) is prime and \( \overline{X_\omega} = V(R/\mathbb{I}_\omega) \).
A Recipe

1. Fix an antidiagonal term order $\rho$ on $R$.
2. Compute the initial ideal $\text{init}_\rho (I_w)$.
3. Take the primary decomposition.

Example:

$I_{2143} = \langle z_{11}, \begin{vmatrix} z_{11} & z_{22} & z_{13} \\ z_{21} & z_{32} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \rangle >$

$\text{init}_\rho (I_{2143}) = \langle z_{11}, z_{31}, z_{22}, z_{13} \rangle$

$= \langle z_{11}, z_{31} \rangle \cap \langle z_{11}, z_{22} \rangle \cap \langle z_{11}, z_{13} \rangle$
Example:

\[ I_{2143} = \langle z_{11}, z_{22}, z_{33} \rangle \]

\[ \text{init}_{2a} (I_{2143}) = \langle z_{11}, z_{31}, z_{22}, z_{13} \rangle \]

\[ = \langle z_{11}, z_{31} \rangle \cap \langle z_{11}, z_{22} \rangle \cap \langle z_{11}, z_{13} \rangle \]

\[ \mathcal{G}_{2143}(x,y) = (x-y_1)(x_3-y_1) + (x-y_1)(x_2-y_2) + (x-y_1)(x_1-y_3) \]
Theorem (Knutson – Miller 2005):

1. Fulton's generators are a Gröbner basis for $I_w$ under any antidiagonal term order.

2. $\text{init}_{\alpha} (I_w)$ is radical.

3. $\text{init}_{\alpha} (I_w) = \bigcap_{P \in \text{Pipes}(w)} I_P$
Each $\overline{X_0}$ defines a class in the $T \times T$ equivariant cohomology of $\text{Mat}(n)$.

To compute $[X]_{T \times T}$ you can use three axioms:

1. Degeneration
2. Additivity
3. Normalization

Theorem (Knutson - Miller 2005):

$$[\overline{X_0}]_{T \times T} = g_0(x; y).$$
Diagonal Degenerations
Knutson, Miller, and Yong (2009) worked out an analogous story for diagonal degenerations of vexillary permutations.

Theorem (KMY 2009):

1. Fulton's generators are a diagonal Gröbner basis if and only if $w$ is vexillary.

2. In this case, primes are indexed by flagged tableaux.
\[ I_{1423} = \langle \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}, \begin{bmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{bmatrix}, \begin{bmatrix} z_{12} & z_{13} \\ z_{22} & z_{23} \end{bmatrix} \rangle \]

\[
\text{initialized}(I_{1423}) = \langle z_{11} z_{22}, z_{11} z_{23}, z_{12} z_{23} \rangle
\]

\[
= \langle z_{11}, z_{12} \rangle \cup \langle z_{11}, z_{23} \rangle \cup \langle z_{22}, z_{23} \rangle
\]
\[ \text{init}_d(\Sigma_{1423}) = \langle z_{11}, z_{12} \rangle \cap \langle z_{11}, z_{23} \rangle \cap \langle z_{22}, z_{23} \rangle \]
Example:

\[ I_{2143} = \langle z_{11} \rangle, \begin{bmatrix}
    z_{11} & z_{22} & z_{33} \\
    z_{21} & z_{22} & z_{23} \\
    z_{31} & z_{32} & z_{33}
\end{bmatrix} \]

\[ = \langle z_{11} \rangle, \begin{bmatrix}
    0 & z_{22} & z_{13} \\
    z_{21} & z_{22} & z_{23} \\
    z_{31} & z_{32} & z_{33}
\end{bmatrix} \]

\[ \text{initial } (I_{2143}) = \langle z_{11}, z_{33}, z_{21}, z_{12} \rangle \]

\[ = \langle z_{11}, z_{33} \rangle \cap \langle z_{11}, z_{21} \rangle \cap \langle z_{11}, z_{12} \rangle \]
Example:

\[ I_{2413} = \langle z_{11}, \begin{vmatrix} \frac{z_{21} z_{22} z_{23}}{z_{31} z_{32} z_{33}} \end{vmatrix} \rangle > \]

\[ = \langle z_{11}, \begin{vmatrix} \frac{z_{21} z_{22} z_{23}}{z_{31} z_{32} z_{33}} \end{vmatrix} \rangle > \]

\[ \text{init}_{\text{alg}}(I_{2143}) = \langle z_{11}, z_{33} z_{21} z_{12} \rangle \]

\[ = \langle z_{11}, z_{33} \rangle \cap \langle z_{11}, z_{21} \rangle \cap \langle z_{11}, z_{12} \rangle \]

\[ \sigma_{2143}(x, y) = (x_1 - y_1)(x_3 - y_3) + (x_1 - y_1)(x_2 - y_1) + (x_1 - y_1)(x_1 - y_2) \]
A Pathology

Caution: initial $(I_w)$ might not be radical!

\[ V(\mathbb{R}/\langle z_{11}, z_{12}, z_{13}, z_{21}, z_{22}, z_{317} \rangle) \text{ shows up in } \text{Spec} \left( \mathbb{R}/\text{initial}(I_{321054}) \right) \text{ with multiplicity!} \]
Conjecture (Hamaker-Pechenik-W. 2020):
BPD's for \( w \) label set theoretic components of \( \text{Spec} \left( R/\text{init}_{\omega} (I_w) \right) \) with multiplicity, i.e. the multiplicity of \( V(R/\mathbb{Z}z_{i,j} : (i,j) \in D^2) \) is
\[
\# \{ p \in \text{BPD}(\omega) : D \text{ indexes the blank tiles in } \mathbb{P}^3 \}.
\]
CDG Generators

\[ Y_{426135} = \langle z_{11}, z_{12}, z_{13}, z_{21}, z_{31} \rangle \]

\[ + \langle 2 \times 2 \text{ minors in } \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & z_{22} & z_{23} & z_{24} \\ 0 & z_{32} & z_{33} & z_{34} \end{bmatrix} \rangle \]

\[ + \langle 3 \times 3 \text{ minors in } \begin{bmatrix} 0 & 0 & 0 & z_{14} & z_{15} \\ 0 & z_{22} & z_{23} & z_{24} & z_{25} \\ 0 & z_{32} & z_{33} & z_{34} & z_{35} \end{bmatrix} \rangle \]

**Question:** When are CDG generators diagonal Gröbner bases?
A permutation is predominant if its Lehmer code is of the form $2^k \lambda$ where $\lambda$ is a partition and $k, n \in \mathbb{Z}_{\geq 0}$.

Example: 426135 has code 313000.

$\lambda = (3,1), \ k = 0, \ n = 3.$
Lemma: $BPD(u) \times BPD(w) \rightarrow BPD(u \cup w)$ is a bijection
We say $w \in S_n$ is banner if we can write

$$w = u_1 u_2 \ldots u_k$$

where the $u_i$'s are (partial) permutations coming from vexillary, predominant, or inverse predominant permutations.
Theorem (Hamaker–Pechenik–W. 2020):
If \( w \) is banner, then

1. The CDG generators are a diagonal Gröbner basis and

2. \( \text{in} \cdot \text{Id}_d (I_w) = \bigcap_{P \in BPD(w)} I_P \).
Theorem (Klein 2020): The CDG generators for \( I_w \) are a diagonal Gröbner basis if and only if \( w \) avoids the patterns:

13254, 21543, 214635, 215364, 241635, 315264, 215634, and 4261735.

(Conjectured by Hamaker - Pechenik - W.)
Theorem (Klein-W. 2021+): If \( \text{init}_{\text{kd}}(I_{w}) \) is radical, \( \text{init}_{\text{kd}}(I_{w}) = \bigcap_{p \in \text{BPD}(w)} I_{p} \).

Corollary: The conjecture holds for CDC permutations.
Transition
Thanks!
A Paradigm of Patterns