Gröbner Geometry of Schubert Polynomials Through Ice

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Based on joint work with Oliver Pechenik and Zachary Hamaker and work in progress with Patricia Klein.

Schubert Polynomials

The complete flag variety $GL(n)/B$ has a special family of subvarieties $\{X_w : w \in S_n\}$ called Schubert varieties.

Each Schubert variety defines a Schubert class $\sigma_w \in H^*(GL(n)/B)$. The Schubert classes are a linear basis for this ring.

By Borel's isomorphism:

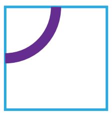
$$H^*(GL(n)/B) \cong \mathbb{Z}[x_1, x_2, \dots, x_n] / I^{S_n}.$$

Lascoux and Schützenberger (1982) defined Schubert polynomials $\{\mathfrak{S}_w(x) : w \in S_n\}$ which are a choice of coset representatives for the images of the Schubert classes under Borel's isomorphism.

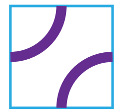
There's also double Schubert polynomials $\{\mathfrak{S}_w(x;y) : w \in S_n\}$ which are enriched versions of single Schuberts and satisfy

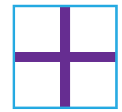
$$\mathfrak{S}_w(x) = \mathfrak{S}_w(x;0).$$

Pipe Dreams

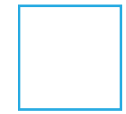


on main antidiagonal

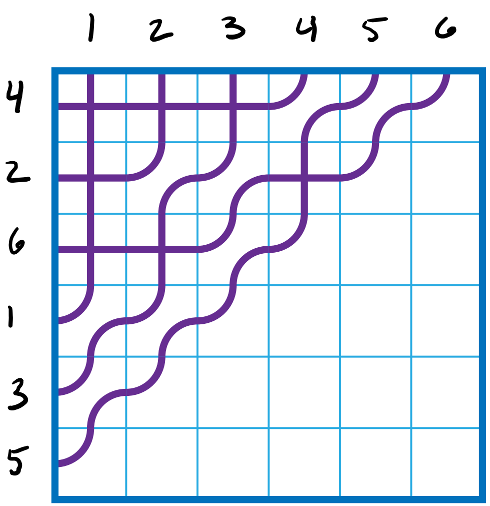




above main antidiagonal



below main antidiagonal



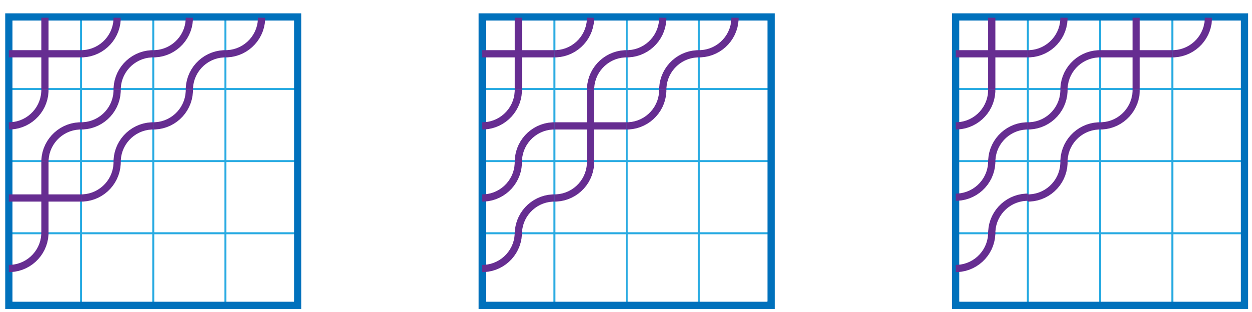
Fill $n \times n$ grid with n pipes that start at the top end at the left and pairwise cross at most one time.

$$wt(P) = (x_1-y_1)(x_1-y_2)(x_1-y_3)(x_2-y_1)(x_2-y_4)(x_3-y_1)(x_3-y_2)$$

Theorem (Fomin-Kirillov 1996 / Bergeron-Billey 1993):

The double Schubert polynomial is

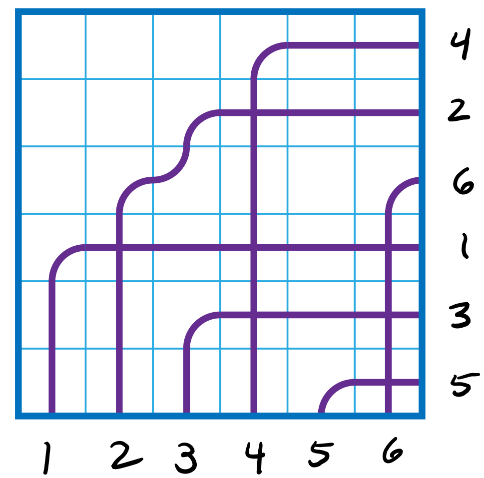
$$\mathfrak{S}_w(x;y) = \sum_{P \in \mathrm{Pipes}(w)} \mathrm{wt}(P).$$

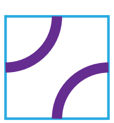


$$\mathfrak{S}_{2143}(x;y) = (x_1-y_1)(x_3-y_1) \quad + \quad (x_1-y_1)(x_2-y_2) \quad + \quad (x_1-y_1)(x_1-y_3)$$

Bumpless Pipe Dreams

Fill $n \times n$ grid with n pipes that start at the bottom, end at the right and pairwise cross at most one time.

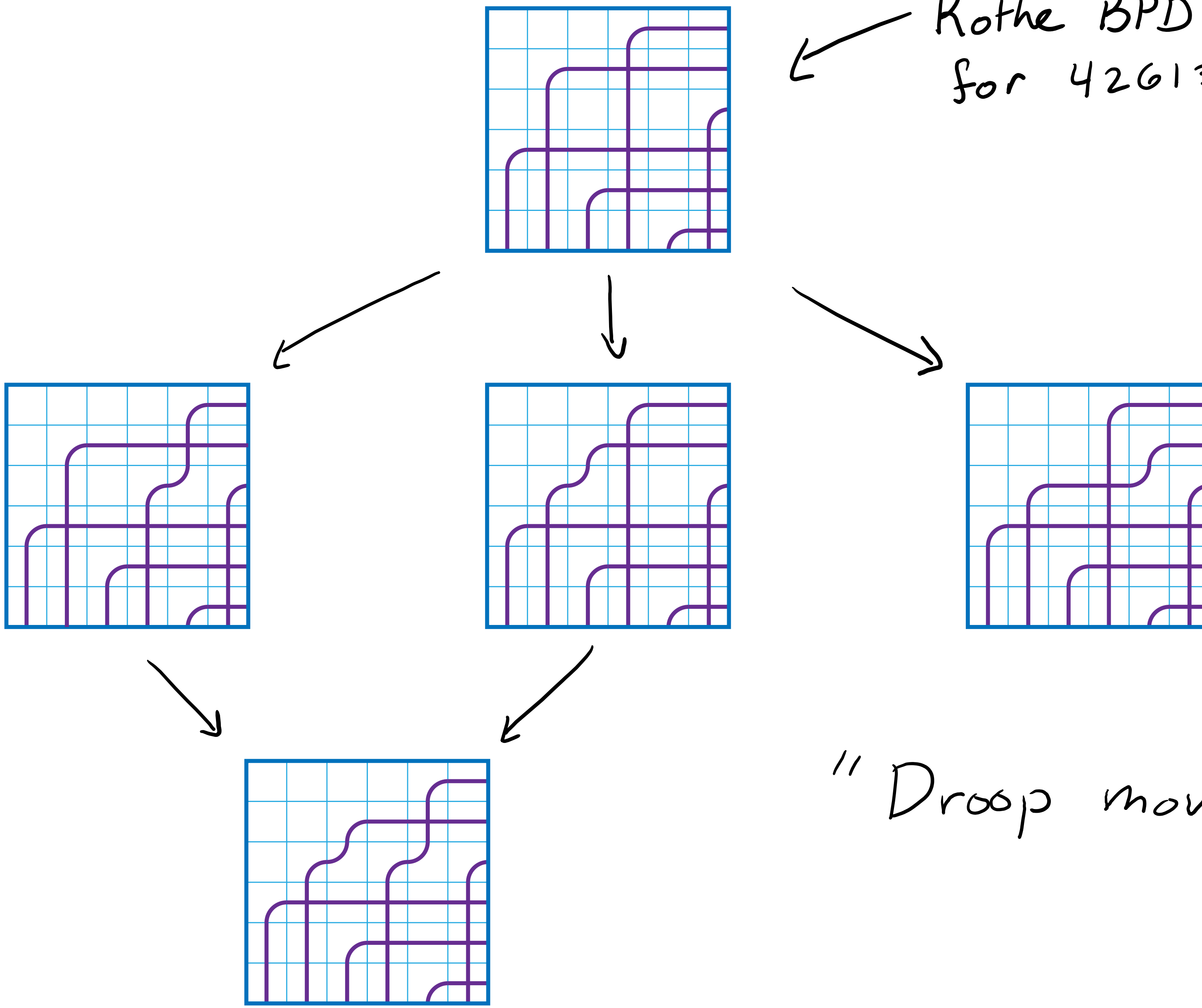




no bumps allowed!

$$wt(P) = (x_1 - y_1)(x_1 - y_2)(x_1 - y_3)$$
$$\cdot (x_2 - y_1)(x_2 - y_2)(x_3 - y_1)(x_3 - y_5)$$

Rothe BPD
for 426135

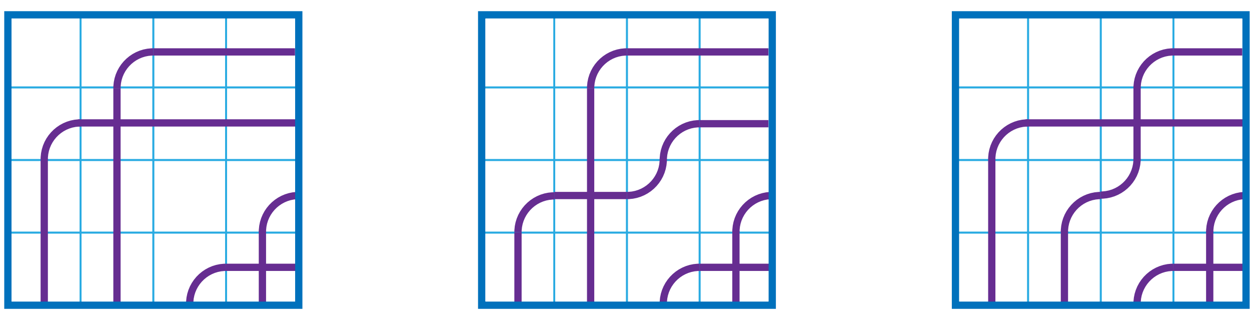


"Droop moves"

Theorem (Lam-Lee-Shimozono 2018):

The double Schubert polynomial is

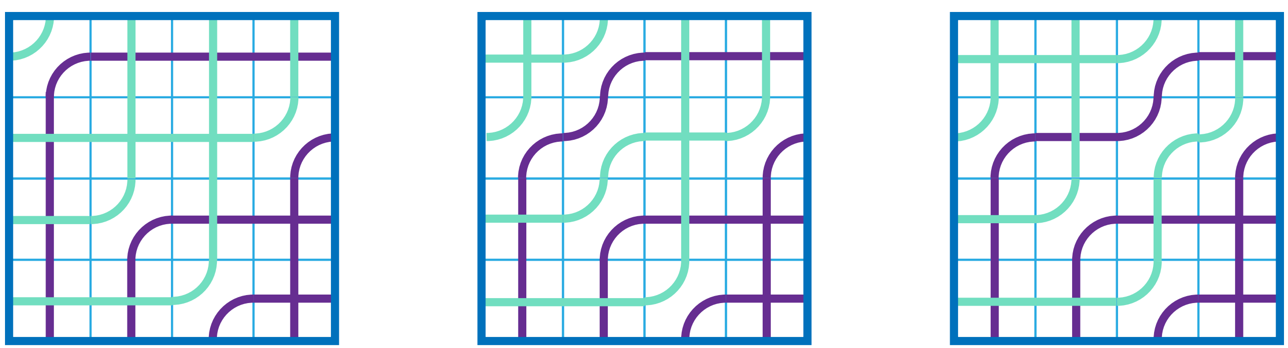
$$\mathfrak{S}_w(x;y) = \sum_{P \in BPD(w)} wt(P).$$



$$\mathfrak{S}_{2143}(x;y) = (x_1-y_1)(x_3-y_3) \quad + \quad (x_1-y_1)(x_2-y_1) \quad + \quad (x_1-y_1)(x_1-y_2)$$

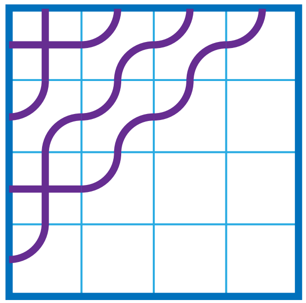
Secretly, BPDs were hiding in an unpublished preprint of Lascoux (2002).

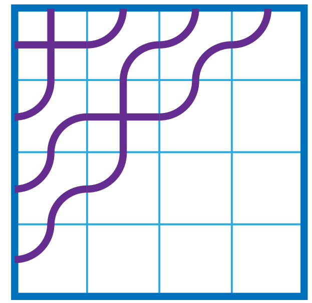
Lascoux gave a formula for double Grothendieck polynomials as a sum over states of the 6-vertex (ice) model.

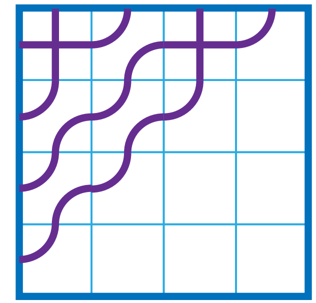


See arXiv:2003.07342.

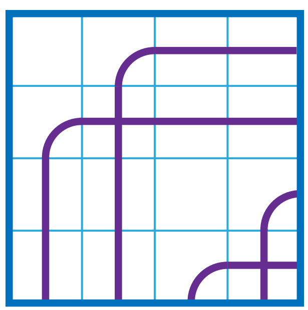
$$\mathfrak{S}_{2143}(x;y) = (x_1-y_1)(x_3-y_1) \quad + \quad (x_1-y_1)(x_2-y_2) \quad + \quad (x_1-y_1)(x_1-y_3)$$

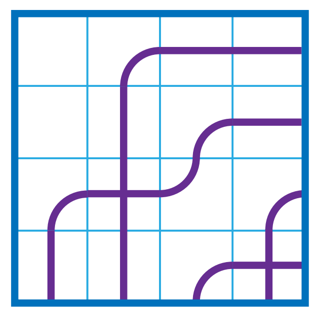


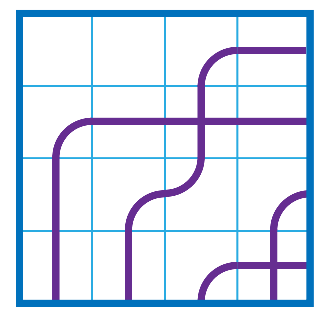




$$\mathfrak{S}_{2143}(x) = x_1 x_3 \quad + \quad x_1 x_2 \quad + \quad x_1^2$$







$$\mathfrak{S}_{2143}(x;y) = (x_1-y_1)(x_3-y_3) \quad + \quad (x_1-y_1)(x_2-y_1) \quad + \quad (x_1-y_1)(x_1-y_2)$$

Matrix Schubert Varieties

Determinantal Ideals

Let $\mathrm{Mat}(m,n)$ be the space of $m \times n$ matrices and

$$Z_{m,n} = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{m1} & z_{m2} & \cdots & z_{mn} \end{bmatrix}$$

a generic matrix.

$R = \mathbb{C}[Z_{m,n}] = \mathbb{C}[z_{11}, z_{12}, \ldots, z_{mn}]$ is the coordinate ring of $\mathrm{Mat}(m,n)$.

Let $I_k(Z_{m,n})$ be the ideal generated by the minors of size k in $Z_{m,n}$

$I_k(Z_{m,n})$ is a determinantal ideal.

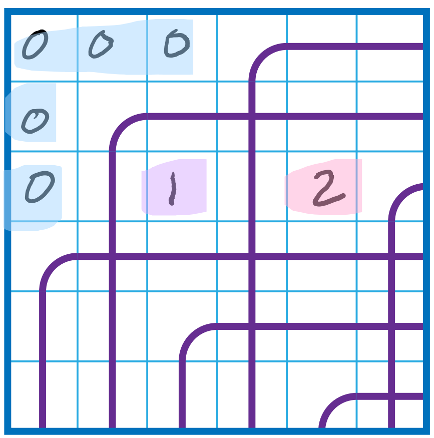
Example:

$$I_2\left(\begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \end{bmatrix}\right) = \left\langle \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}, \begin{vmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{vmatrix}, \begin{vmatrix} z_{12} & z_{13} \\ z_{22} & z_{23} \end{vmatrix} \right\rangle$$

$V(R/I_{k+1}(Z_{m,n}))$ is the set of matrices $M \in Mat(m,n)$ such that $\mathrm{rank}(M) \leq k$.

Schubert Determinantal Ideals

Given $w \in S_n$, the Schubert determinantal ideal $I_w \subseteq R = \mathbb{C}[z_{n,n}]$ is generated by minors of varying size, determined by w.



$$I_{426135} = \langle z_{11}, z_{12}, z_{13}, z_{21}, z_{31} \rangle$$

$$+ \left\langle 2\times 2 \text{ minors in } \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix} \right\rangle$$

$$+ \left\langle 3\times 3 \text{ minors in } \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} \\ z_{31} & z_{32} & z_{33} & z_{34} & z_{35} \end{bmatrix} \right\rangle$$

Fulton's Generators

Matrix Schubert Varieties

$$\begin{tikzcd} GL(n) \arrow[r, hook, "i"] \arrow[d, two heads, "\pi"'] & Mat(n) \\ GL(n)/B \end{tikzcd}$$

Flag variety

$$\begin{tikzcd} \pi^{-1}(X_w) \arrow[r, hook, "i"] \arrow[d, two heads] & \overline{i(\pi^{-1}(X_w))} \\ X_w \end{tikzcd}$$

Matrix Schubert variety

Schubert variety

Write $\overline{X_w} := \overline{i(\pi^{-1}(X_w))}$.

Theorem (Fulton 1992): I_w is prime and

$\overline{X_w} = V(R/I_w)$.

A Recipe

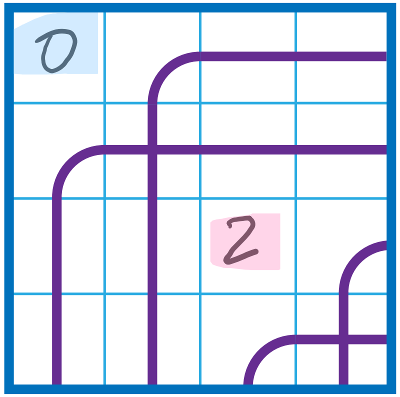
① Fix an antidiagonal term order $<_a$ on R.

② Compute the initial ideal $\mathrm{init}_{<_a}(I_w)$.

③ Take the primary decomposition.

Example:

$$I_{2143} = \left\langle z_{11},\ \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \right\rangle$$

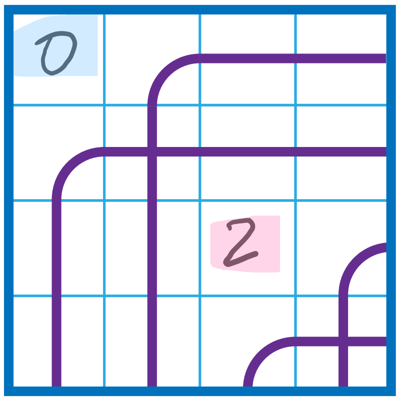


$$\mathrm{init}_{<_a}(I_{2143}) = \langle z_{11},\ z_{31} z_{22} z_{13} \rangle$$

$$= \langle z_{11}, z_{31} \rangle \cap \langle z_{11}, z_{22} \rangle \cap \langle z_{11}, z_{13} \rangle$$

Example:

$$I_{2143} = \left\langle z_{11}, \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \right\rangle$$



$$\mathrm{init}_{<_a}(I_{2143}) = \langle z_{11}, z_{31} z_{22} z_{13} \rangle$$

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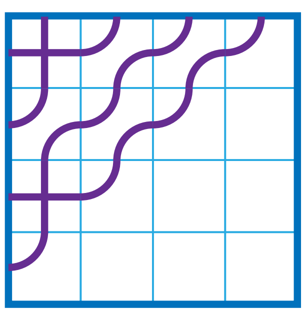
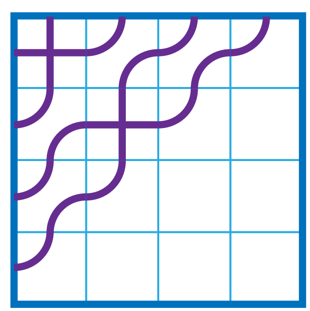
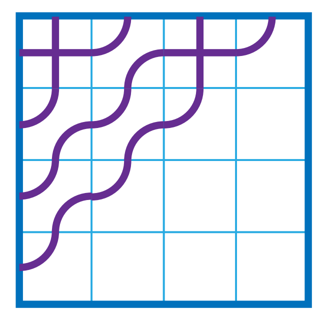






$$\mathfrak{G}_{2143}(x;y) = (x_1-y_1)(x_3-y_1) + (x_1-y_1)(x_2-y_2) + (x_1-y_1)(x_1-y_3)$$

Theorem (Knutson - Miller 2005):

1. Fulton's generators are a Gröbner basis for I_w under any antidiagonal term order.

2. $\mathrm{init}_{<_a}(I_w)$ is radical.

3. $$\mathrm{init}_{<_a}(I_w) = \bigcap_{P \in \mathrm{Pipes}(w)} I_P$$

But wait, there's more!

Each $\overline{X_w}$ defines a class in the $T \times T$ equivariant cohomology of $Mat(n)$.

To compute $[X]_{T \times T}$ you can use three axioms

1. Degeneration
2. Additivity
3. Normalization.

Theorem (Knutson - Miller 2005):

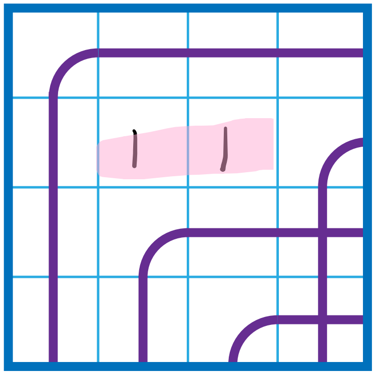
$$[\overline{X_w}]_{T \times T} = \mathfrak{G}_w(x;y).$$

Diagonal Degenerations

Knutson, Miller, and Yong (2009) worked out an analagous story for diagonal degenerations of vexillary permutations.

Theorem (KMY 2009):

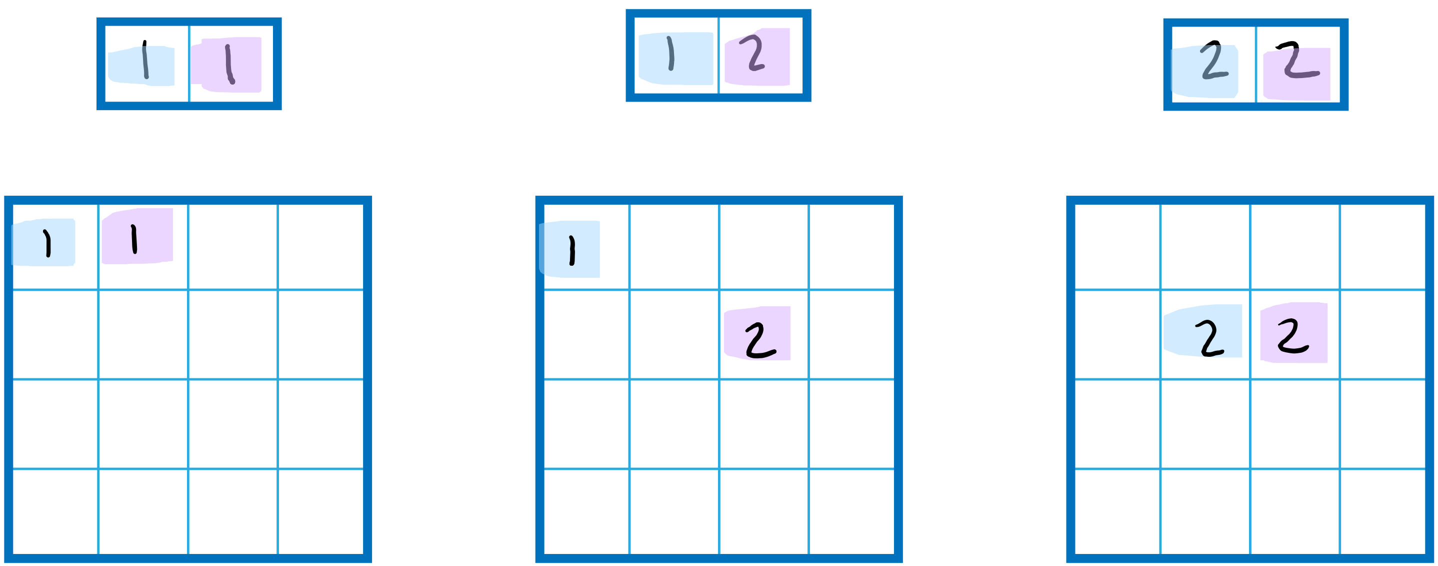
1. Fulton's generators are a diagonal Gröbner basis if and only if w is vexillary.
2. In this case, primes are indexed by flagged tableaux.



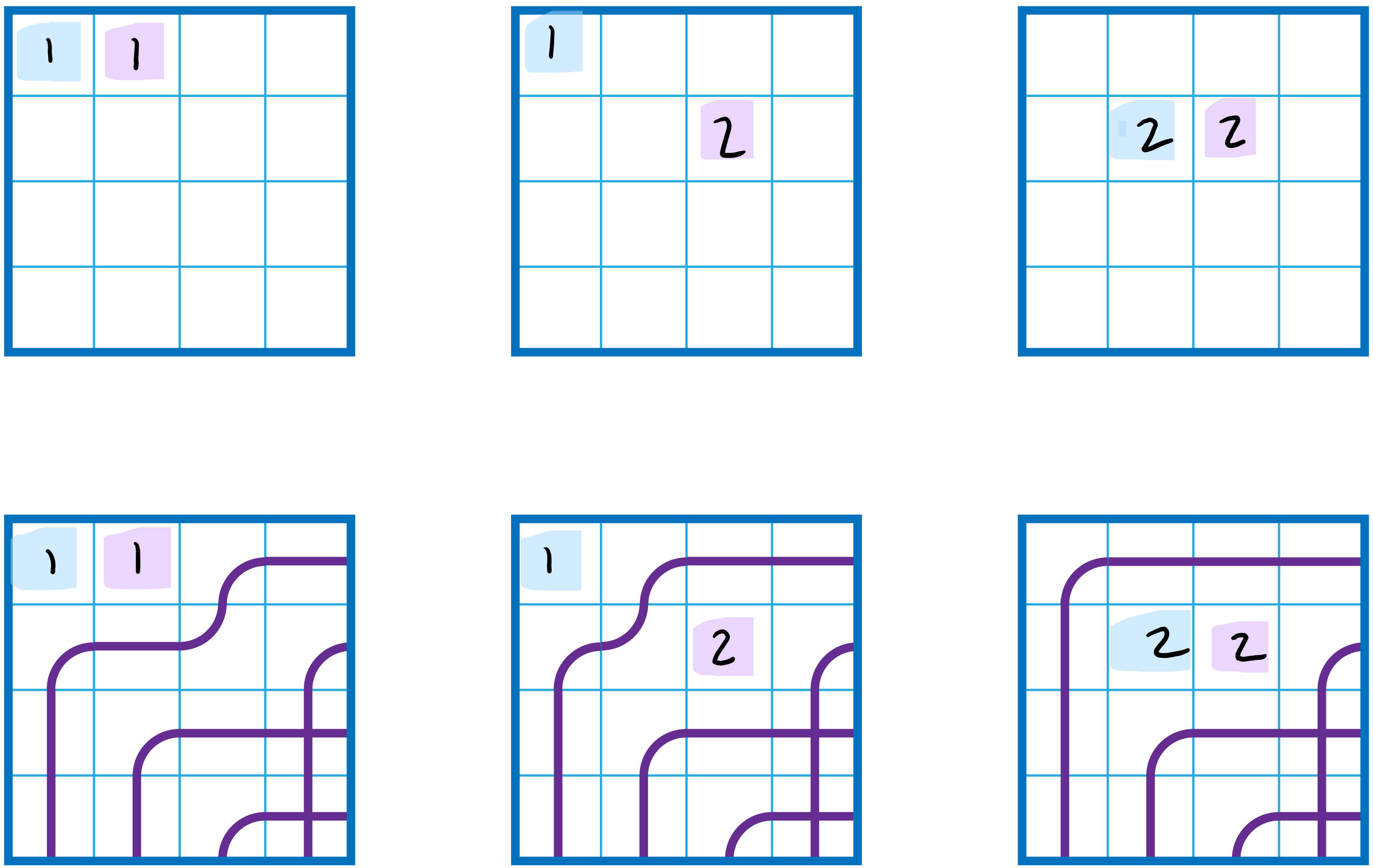
$$I_{1423} = \left\langle \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}, \begin{vmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{vmatrix}, \begin{vmatrix} z_{12} & z_{13} \\ z_{22} & z_{23} \end{vmatrix} \right\rangle$$

$$\mathrm{init}_{<_d}(I_{1423}) = \langle z_{11} z_{22}, z_{11} z_{23}, z_{12} z_{23} \rangle$$

$$= \langle z_{11}, z_{12} \rangle \cap \langle z_{11}, z_{23} \rangle \cap \langle z_{22}, z_{23} \rangle$$

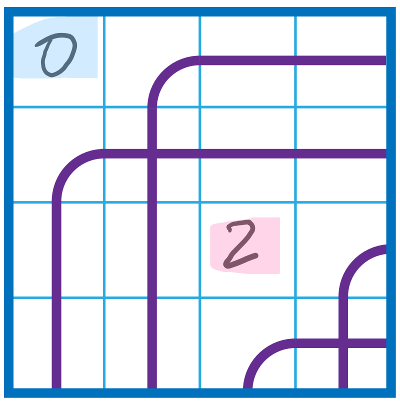


$$\mathrm{init}_{<_d}(I_{1423}) = \langle z_{11}, z_{12} \rangle \cap \langle z_{11}, z_{23} \rangle \cap \langle z_{22}, z_{23} \rangle$$



Example:

$$I_{2143} = \left\langle z_{11},\ \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \right\rangle$$



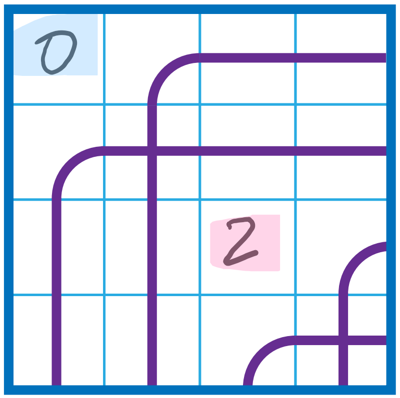
$$= \left\langle z_{11},\ \begin{vmatrix} 0 & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \right\rangle$$

$$\mathrm{init}_{<_d}(I_{2143}) = \langle z_{11},\ z_{33} z_{21} z_{12} \rangle$$

$$= \langle z_{11}, z_{33} \rangle \cap \langle z_{11}, z_{21} \rangle \cap \langle z_{11}, z_{12} \rangle$$

Example:

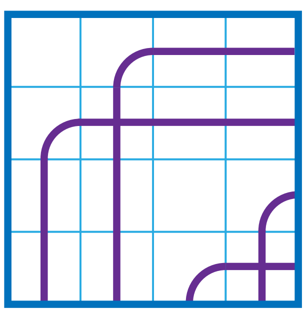
$$I_{2143} = \left\langle z_{11},\ \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \right\rangle$$

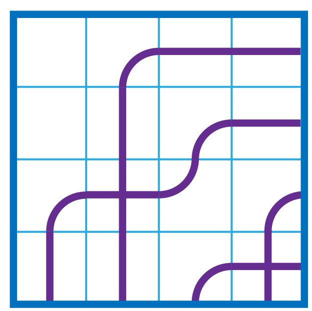


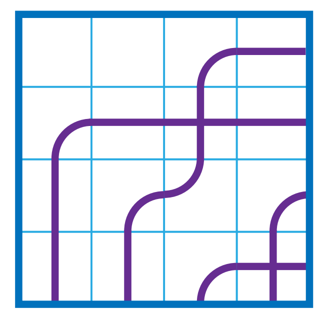
$$= \left\langle z_{11},\ \begin{vmatrix} 0 & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \right\rangle$$

$$\mathrm{init}_{<_d}(I_{2143}) = \langle z_{11},\ z_{33} z_{21} z_{12} \rangle$$

$$= \langle z_{11}, z_{33} \rangle \cap \langle z_{11}, z_{21} \rangle \cap \langle z_{11}, z_{12} \rangle$$



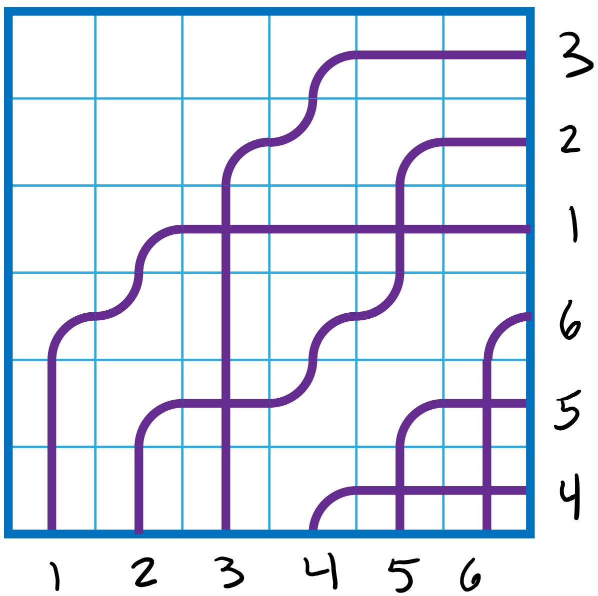


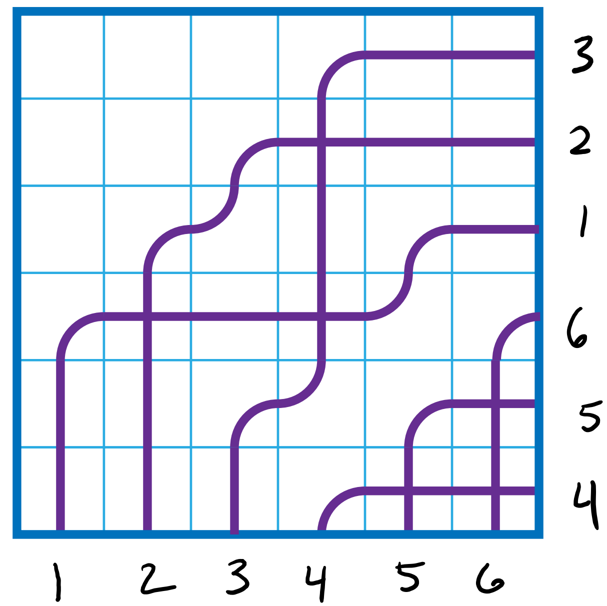


$$\mathfrak{G}_{2143}(x;y) = (x_1 - y_1)(x_3 - y_3) \quad + \quad (x_1 - y_1)(x_2 - y_1) \quad + \quad (x_1 - y_1)(x_1 - y_2)$$

A Pathology

Caution: $\mathrm{init}_{<_d}(I_w)$ might not be radical!





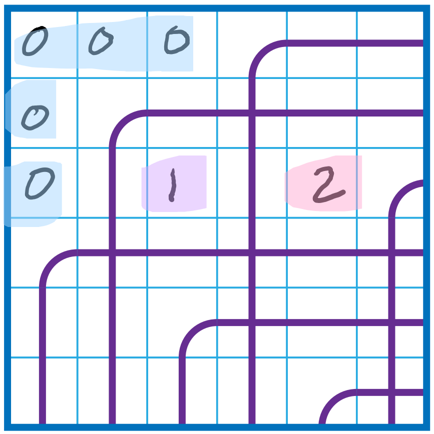
$V(R/\langle z_{11}, z_{12}, z_{13}, z_{21}, z_{22}, z_{31}\rangle)$ shows up in $\mathrm{Spec}(R/\mathrm{init}_{<_d}(I_{321654}))$ with multiplicity!

Conjecture (Hamaker-Pechenik-W. 2020):

BPD's for w label set theoretic components of $\mathrm{Spec}(R/\mathrm{init}_{\mathrm{ed}}(I_w))$ with multiplicity, i.e. the multiplicity of $V(R/\langle z_{ij} : (i,j) \in D \rangle)$ is

$$\#\{P \in BPD(w) : D \text{ indexes the blank tiles in } P\}.$$

CDG Generators



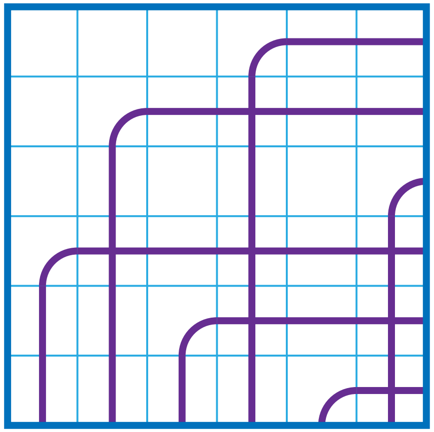
$$I_{426135} = \langle z_{11}, z_{12}, z_{13}, z_{21}, z_{31} \rangle$$

$$+ \left\langle \text{2x2 minors in } \begin{bmatrix} 0 & 0 & 0 \\ 0 & z_{22} & z_{23} \\ 0 & z_{32} & z_{33} \end{bmatrix} \right\rangle$$

$$+ \left\langle \text{3x3 minors in } \begin{bmatrix} 0 & 0 & 0 & z_{14} & z_{15} \\ 0 & z_{22} & z_{23} & z_{24} & z_{25} \\ 0 & z_{32} & z_{33} & z_{34} & z_{35} \end{bmatrix} \right\rangle$$

Question: When are CDG generators diagonal Gröbner bases?

Predominant Permutations

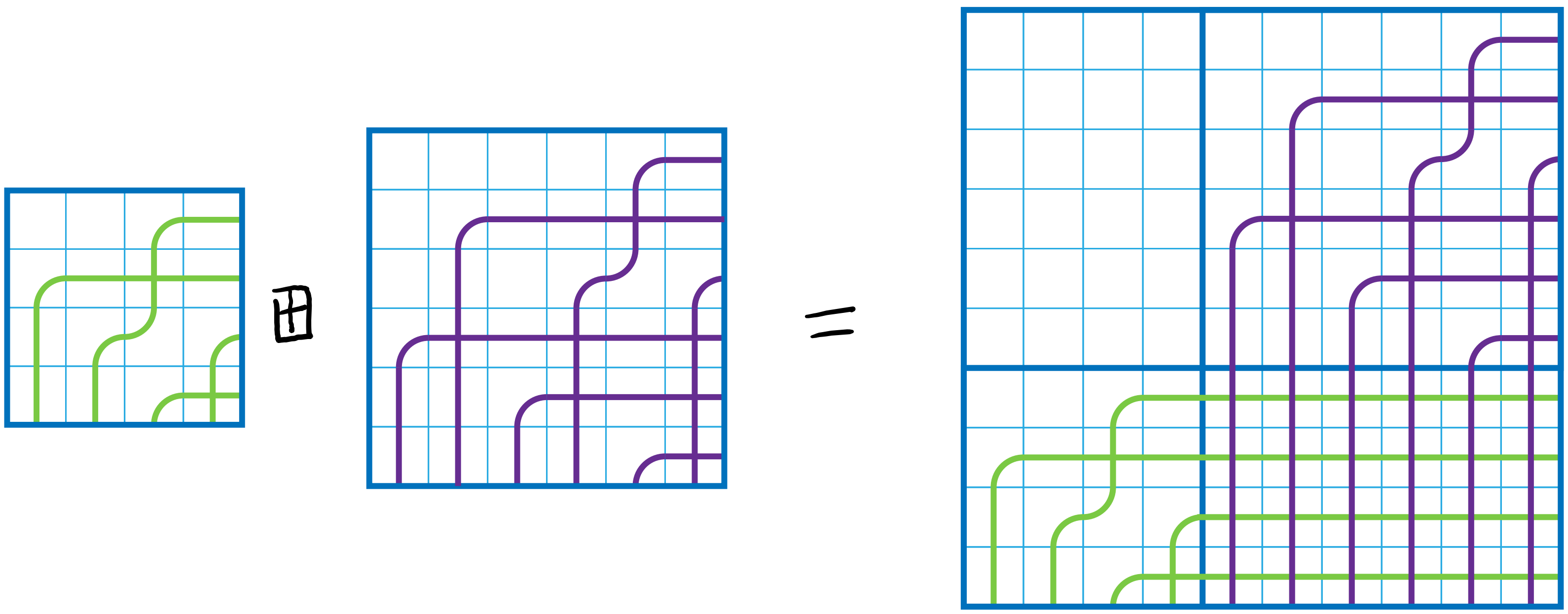


A permutation is predominant if its Lehmer code is of the form $\lambda 0^k n$ where λ is a partition and $k, n \in \mathbb{Z}_{\geq 0}$.

Example: 426135 has code 313000.

$\lambda = (3,1)$, $k = 0$, $n = 3$.

Skew Sums of BPDS



Lemma: $BPD(u) \times BPD(w) \xrightarrow{\boxplus} BPD(u \boxplus w)$ is a bijection

Banner Permutations

We say $w \in S_n$ is banner if we can write

$w = u_1 \boxplus u_2 \boxplus \cdots \boxplus u_k$ where the u_i's are (partial) permutations coming from vexillary, predominant, or inverse predominant permutations.

Theorem (Hamaker–Pechenik–W. 2020):

If w is banner, then

1. The CDG generators are a diagonal Gröbner basis and

2. $$\mathrm{init}_{<_d}(I_w) = \bigcap_{P \in BPD(w)} I_P .$$

Theorem (Klein 2020): The CDG generators for I_w are a diagonal Gröbner basis if and only if w avoids the patterns:

13254, 21543, 214635, 215364, 241635,

315264, 215634, and 4261735.

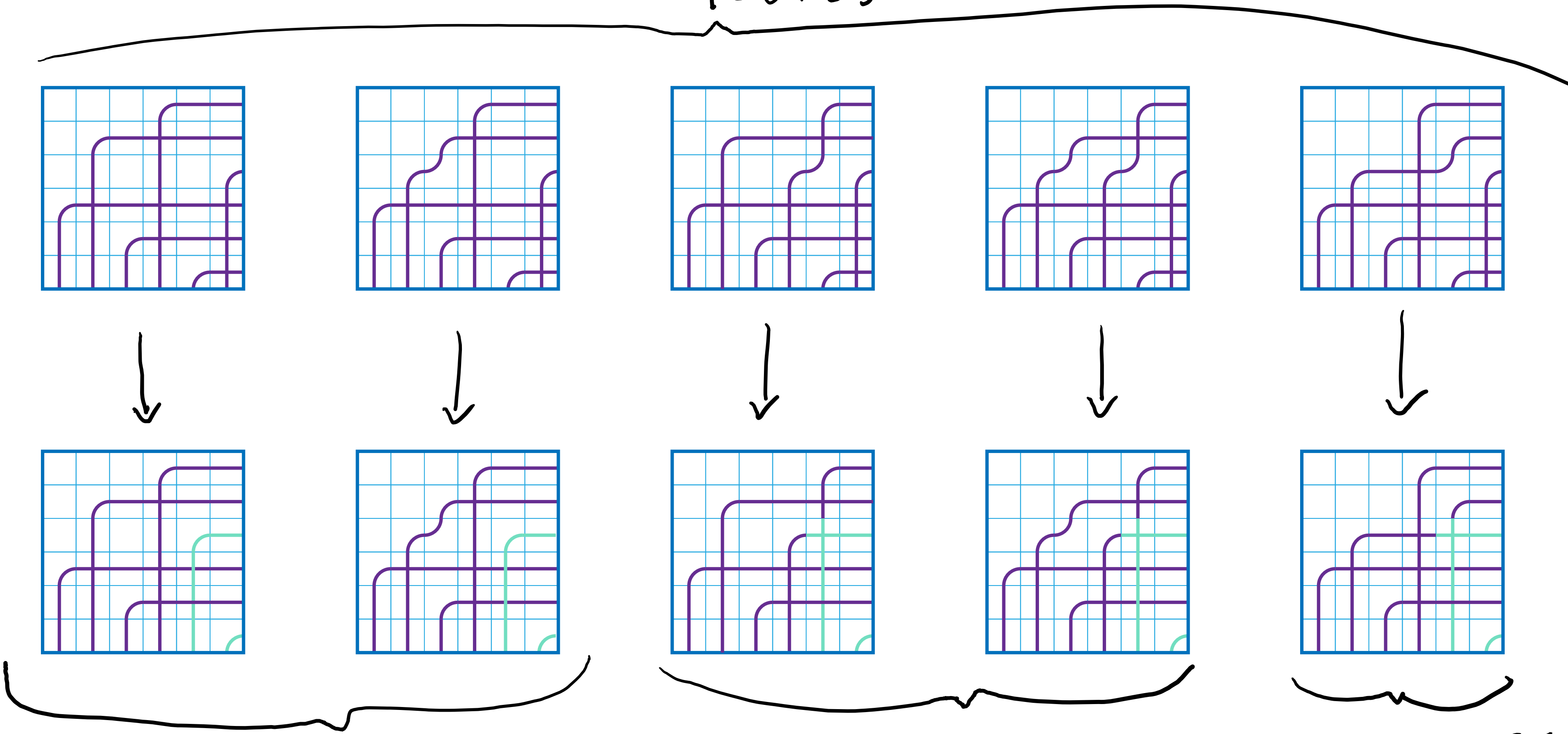
(Conjectured by Hamaker-Pechenik-W.)

Theorem (Klein–W. 2021+): If $\mathrm{init}_{<_\alpha}(I_w)$ is radical, $\mathrm{init}_{<_\alpha}(I_w) = \bigcap_{P \in BPD(w)} I_P$.

Corollary: The conjecture holds for CDG permutations.

Transition

426135

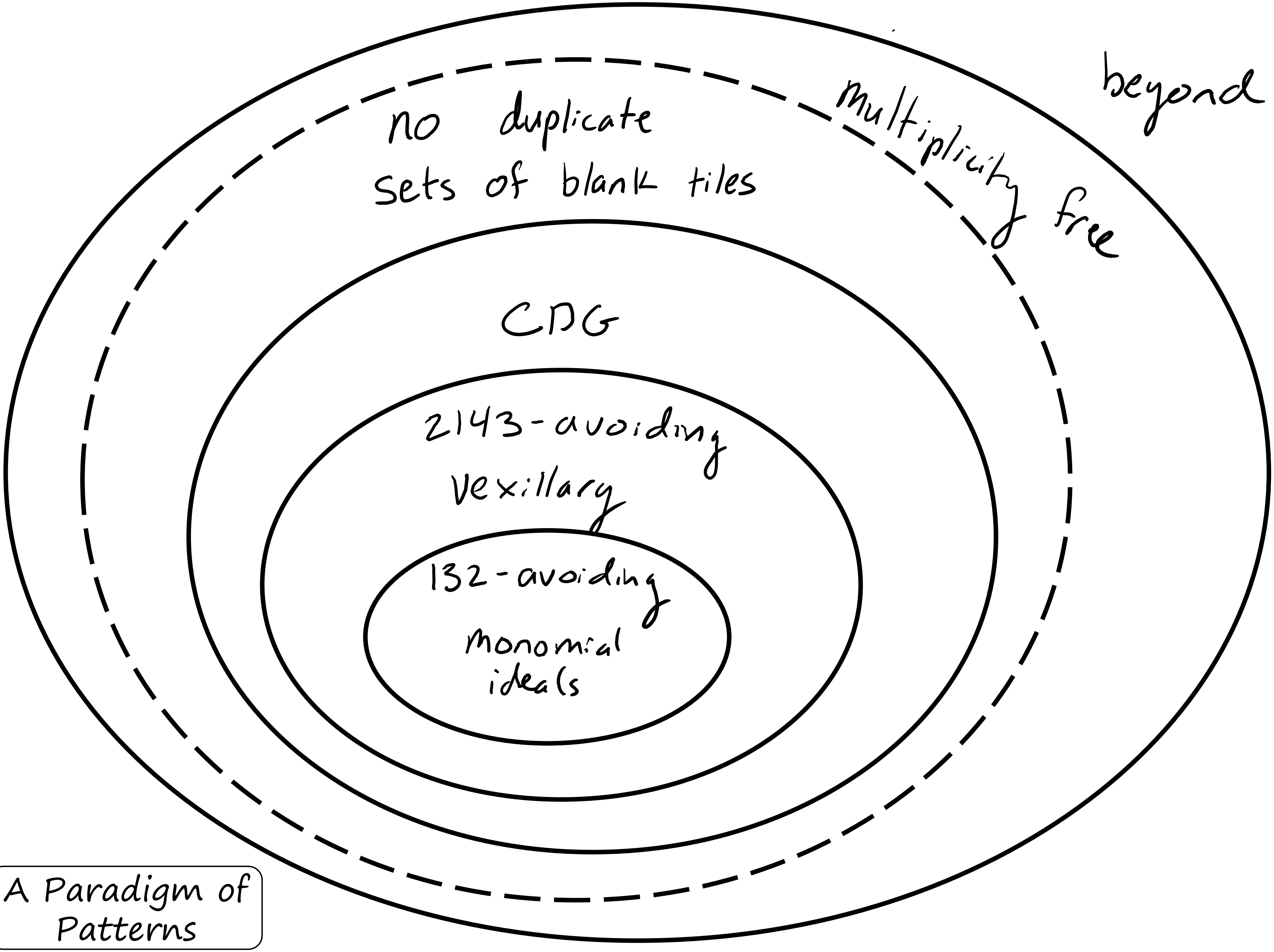


425136

524136

452136

Thanks!



A Paradigm of Patterns