Session 1

Tuesday, March 23

Time	Speaker
10:45 -10:55	Semin Yoo, University of Rochester
10:55 - 11:05	Michael Perlman, Queen's University
11:05 - 11:15	Papri Dey, University of Missouri-Columbia
11:15 - 11:25	Aram Bingham, Tulane University of Louisiana
11:25 - 11:35	Yifeng Huang, University of Michigan

Combinatorics of quadratic spaces over finite fields

Semin Yoo

University of Rochester

Lightning Talk
Geometry and Combinatorics from Root Systems
March 23, 2021

Motivation

Recall. q - binomial coefficients

$$\binom{n}{k}_q = \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})} = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}$$

$$= \text{the number of } k\text{-dimensional subspaces of } \mathbb{F}_q^n.$$

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$$= \text{the number of } k\text{-dimensional subspaces of } \mathbb{F}_q^n.$$

	Field with one element	$\mathbb{F}_q$ **(q-analogues)**
object	$[n]=\{1,2,\cdots,n\}$	$\mathbb{F}_q^n$
subobject	a k set in $[n]$	a k-dimensional subspace of $\mathbb{F}_q^n$
bracket	n	the number of lines in $\mathbb{F}_q^n$
factorial	$n!$	$[n]_q!$
poset	B_n	$L_n(q)$
group	$\lvert S_n\rvert = n!$	$\lvert GL(n,q)\rvert = q^{n(n-1)/2}(q-1)^n\,[n]_q!$
flag	flags in $[n]$	flags in $\mathbb{F}_q^n$
binomial coefficient	$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \left\lvert \frac{S_n}{S_k \times S_{n-k}} \right\rvert$	$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![(n-k)]_q!} = \left\lvert \frac{GL(n,q)}{\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}} \right\rvert$
connection	$\lim_{q\to 1}\binom{n}{k}_q = \binom{n}{k}$	

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- Let us call k-dimensional quadratic subspace $W \subset (\mathbb{F}_q^n, Q)$ a **$\mathrm{dot}_{\mathrm{k}}$-subspace** if $(W, Q|_W) \simeq (\mathbb{F}_q^k, \mathrm{dot}_k)$.

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- We define **dot-analogues** as follows:
  $[k]_d := |\mathrm{dot}_1\text{-subspaces in } \mathrm{dot}_k|$; $[n]_d! := [n]_d \cdots [1]_d$;
  $\binom{n}{k}_d = \frac{[n]_d!}{[k]_d![(n-k)]_d!}$.

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- $\binom{n}{k}_d < \binom{n}{k}_q$.

Main Goal: Study related combinatorics of $\binom{n}{k}_d$ and its applications.

Theorem (Y., 2019, 2020+)

- $\binom{n}{k}_d$ *can be written by the q-binomial coefficients.*
  For example, when $q \equiv 1 \pmod 4$*, and* n, k *are odd,*

$$\binom{n}{k}_d = \frac{q^{\frac{k(n-k)}{2}}(q^{\frac{n-1}{2}}+1)(q^{\frac{n-1}{2}}-1)(q^{\frac{n-3}{2}}+1)\cdots(q^{\frac{n-k+2}{2}}-1)(q^{\frac{n-k}{2}}+1)}{2(q^{\frac{k-1}{2}}+1)(q^{\frac{k-1}{2}}-1)(q^{\frac{k-3}{2}}+1)\cdots(q-1)\cdot 1}$$
$$= \frac{1}{2}q^{\frac{k(n-k)}{2}}(q^{\frac{n-k}{2}}+1)\binom{\frac{n-1}{2}}{\frac{k-1}{2}}_{q^2}$$

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- $|O(n,q)| = 2^n\,[n]_d!$.

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- $\binom{n}{k}_d$ *are polynomials of degree* $k(n-k)$ *in* q.
- $|O(n,q)| = 2^n\,[n]_d!$.
- $\binom{n}{k}_d = \left|\frac{O(n,q)}{O(k,q)\times O(n-k,q)}\right| = |Gr_d(n,k)| < |Gr_q(n,k)| = \binom{n}{k}_q$.

	q-analogues	dot-analogues
space	$\mathbb{F}_q^n$	$(\mathbb{F}_q^n, \mathrm{dot}_n)$
subspace	a k-dimensional subspace of $\mathbb{F}_q^n$	a dot_k-subspace of dot_n
bracket	the number of lines in $\mathbb{F}_q^n$	the number of spacelike lines in $(\mathbb{F}_q^n, \mathrm{dot}_n)$
factorial	$[n]_q!$	$[n]_d!$
poset	$L_n(q)$	$E_n(q)$
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Table: The q-analogues and the dot-analogues (Y., 2019+).

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Table: The q-analogues and the dot-analogues (Y., 2019+).

Question.

$$\begin{tikzcd} ?? \arrow[r, dash] \arrow[d, dash] & \binom{n}{k}_d \arrow[d, dash] \\ \binom{n}{k} \arrow[r, dash] & \binom{n}{k}_q \end{tikzcd} .$$

Recall $\lim_{q\to 1}\binom{n}{k}_q = \binom{n}{k}$ gives a connection between $\binom{n}{k}_q$ and $\binom{n}{k}$.

Question. $\lim_{q\to 1}\binom{n}{k}_d = ?$

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Definition

A set A is called **symmetric** in $\mathbb{Z}/(n+1)\mathbb{Z}$ if $A = -A$ and $0 \notin A$.

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A set A is called **symmetric** in $\mathbb{Z}/(n+1)\mathbb{Z}$ if $A = -A$ and $0 \notin A$.

Theorem (Y., 2020+)

$\lim_{q\to \pm 1} \binom{n}{k}_d$ *is the number of symmetric k-sets in $\mathbb{Z}/(n+1)\mathbb{Z}$.*

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Questions.

- Combinatorial descriptions of $\binom{n}{k}_d$?
- Analogues of binomial theorem?

References

Pete L. Clark
Quadratic forms chapter I: Witt's theory
http://math.uga.edu/~pete/quadraticforms.pdf

Semin Yoo (2019+)
Combinatorics of quadratic spaces over finite fields
Preprint

Thank you for your attention!

MIXED HODGE STRUCTURE ON LOCAL COHOMOLOGY WITH SUPPORT IN DETERMINANTAL VARIETIES

MICHAEL PERLMAN
(QUEEN'S UNIVERSITY)
MP174@QUEENSU.CA

SETTING:

- X = smooth complex variety
- $Z \subseteq X$ closed subvariety
- $\mathcal{H}^j_Z(\mathcal{O}_X)$ = local cohomology with support in Z

EXAMPLE: Z = hypersurface

$$\mathcal{H}^1_Z(\mathcal{O}_X) = \frac{\bigcup_{k\geq 0} \mathcal{O}_X(kZ)}{\mathcal{O}_X}$$

GOAL: Understand how $\mathcal{H}^j_Z(\mathcal{O}_X)$ detects/measures singularities of Z.

FACT: They are mixed Hodge modules (Saito)

$\Rightarrow$ two increasing filtrations

1. Hodge filtration $F_\bullet$, infinite filtration coherent $\mathcal{O}_X$-modules (log resolutions)

2. Weight filtration $W_\bullet$, finite filtration by D_X-modules, semi-simple quotients

SPECIAL CASE: Z = hypersurface

$$F_k \mathcal{H}^1_Z(\mathcal{O}_X) \Longleftrightarrow \text{Hodge ideals } I_k(Z) \subseteq \mathcal{O}_X$$

(Mustață-Popa)

Detect smooth, log canonical, rational sing

$$\Updownarrow \qquad\qquad \Updownarrow \qquad\qquad \Uparrow$$

$$\left(\underset{\text{all } k}{I_k(Z)=\mathcal{O}_X}\right), \quad \left(I_0(Z)=\mathcal{O}_X\right), \quad \left(I_1(Z)=\mathcal{O}_X\right)$$

THM (P.-Raicu '20): $Z = V(\det) \subseteq \mathbb{C}^{n\times n}$.

$$I_K(Z) = \bigcap_{p=1}^{n-1} J_p^{\left((n-p)(K-1) - \binom{n-p}{2}\right)}$$

symbolic power

determinantal ideal $p\times p$ minors

REMARKS:

- $I_0(Z) = I_1(Z) = \mathcal{O}_X \Rightarrow$ rational sing
- $I_2(Z) = J_{n-1}$

TECHNIQUES:

- Rep theory $GL_n(\mathbb{C})$, equivariant D-modules
- Do it WITHOUT log resolution

THM (P.-Raicu '20, P. '21): $Z_p = \{\text{rank} \leq p\} \subseteq \mathbb{C}^{m \times n}$.

Calculate

$$F_\bullet \mathcal{H}^\bullet_{Z_p}(\mathcal{O}_X), \quad W_\bullet \mathcal{H}^\bullet_{Z_p}(\mathcal{O}_X).$$

Weight of $\mathcal{D}_X$-simple factor depends only on its support and cohom. deg.

FUTURE WORK:

① Hodge ideals, mixed Hodge structure on other spaces with finitely many orbits

② Connections with motivic Chern classes (Brasselet-Schürmann-Yokura).

- Determinantal case known: [Fehér-Rimányi-Weber '19]

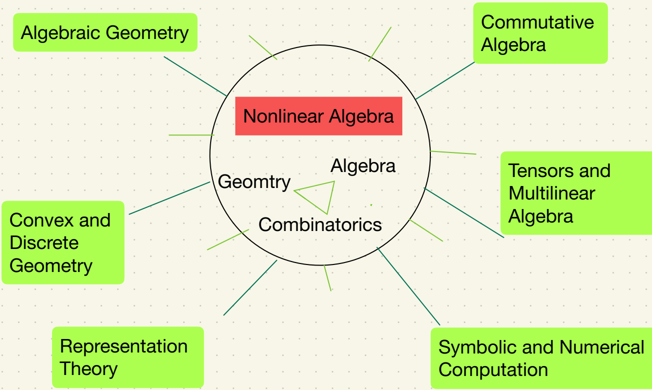
Geometric and Combinatorial aspects of Nonlinear Algebra

Papri Dey

Department of Mathematics
University of Missouri, Columbia

ICERM Fall 2021 CAG:March 23,
Lightning Talk

University of Missouri, Columbia



Three Objectives

1. Computational Aspects
2. Applications
3. Develop mathematical theory and algorithms

Algebraic Torus $(\mathbb{C}^*)^n \leftrightarrow \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$ Laurent Polynomials

Notation: $\mathbf{x}^\alpha \to x_1^{\alpha_1} \dots x_n^{\alpha_n}, \ \alpha = (\alpha_1 \dots \alpha_n) \in \mathbb{Z}^n$

Newton Polytope of $f(\mathbf{x}) = \sum_\alpha c_\alpha \mathbf{x}^\alpha \in \mathbb{C}[\mathbf{x}^{\pm}]$: The convex hull of the finite set $\{\alpha : c_\alpha \neq 0\}$.

Convex polytope with vertices in $\mathbb{Z}^n$.

$\mathcal{A} = \{\alpha_0, \dots, \alpha_s\} \subset \mathbb{Z}^n \to \mathcal{L}_\mathcal{A} = \{f(\mathbf{x}) = \sum_{i=0}^s c_i \mathbf{x}^{\alpha_i} : \alpha_i \in \mathcal{A}, c_i \in \mathbb{C}, \forall i\}$

The convex hull of $\mathcal{A}$ is the Newton polytope of a generic element of $\mathcal{L}_\mathcal{A}$, denoted as $\triangle_\mathcal{A}$.

BKK

For a generic choice of $f_1, \dots, f_n \in \mathcal{L}_\mathcal{A}$, the number of solutions in $(\mathbb{C}^*)^n$ of the system $f_1(\mathbf{x}) = \cdots = f_n(\mathbf{x}) = 0$ is the same, and is equal to $n!\mathrm{vol}(\triangle_\mathcal{A})$

Toric variety and Convex polytope

Due to Khovanskii...one of the references is Escobar and Kaveh 2020...
Consider the map $\psi_{\mathcal{A}} : (\mathbb{C}^*)^n \to \mathbb{CP}^s$ such that $\mathbf{x} \mapsto (\mathbf{x}^{\alpha_0} : \cdots : \mathbf{x}^{\alpha_s})$,

$$T_{\mathcal{A}} := \psi_{\mathcal{A}}(\mathbb{C}^*)^n \cong (\mathbb{C}^*)^n, \text{provided the differences of elements of } \mathcal{A} \text{ generate } \mathbb{Z}^n$$

The toric variety $X_{\mathcal{A}}$ is the closure of the image of the map $\psi_{\mathcal{A}}$ in $\mathbb{CP}^s$.

- The torus $(\mathbb{C}^*)^n$ acts on $\mathbb{CP}^s$ by $\mathbf{x}(z_0 : \cdots : z_s) = (\mathbf{x}^{\alpha_0} z_0 : \cdots : \mathbf{x}^{\alpha_s} z_s)$
- the variety $X_{\mathcal{A}}$ is the closure of the orbit of $(1 : \cdots : 1)$.

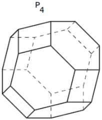
The degree of $X_{\mathcal{A}} \subset \mathbb{CP}^s$ is equal to $n!vol(\triangle_{\mathcal{A}})$

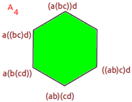
Moment Map: $\mu_{\mathcal{A}} : X_{\mathcal{A}} \to \triangle_{A}$.
$\mu_{\mathcal{A}} : \mathbb{CP}^s \to \mathbb{R}^n$ is defined by

$$(z_0 : \cdots : z_s) \mapsto \sum_{i=0}^{s} \left(\frac{|z_i|^2}{\sum_{j=0}^{s} |z_j|^2} \right) \alpha_i \in \triangle_{\mathcal{A}}$$

- $\mu_{\mathcal{A}}$ is invariant under the action of $(\mathbb{C}^*)^n$
- $\mu_{\mathcal{A}}(\mathbb{CP}^s) = \mu_{\mathcal{A}}(X_{\mathcal{A}}) = \triangle_{\mathcal{A}}$

- The *flag variety* F_n: $\{0\} = V_0 \subsetneq V_1 \subsetneq \dots V_n = \mathbb{C}^n \leftrightarrow M \in GL(n, \mathbb{C}), V_i =$ the row span of the top i rows of M
- **Plucker Coordinates of the subspace V_i:** $(p_I(M) : |I| = i) \in \mathbb{CP}^{\binom{n}{i}-1}, 1 \leq i \leq n$
- F_n: A projective variety in $\mathbb{CP}^{\binom{n}{1}-1} \times \cdots \times \mathbb{CP}^{\binom{n}{n-1}-1}$
- The torus $(\mathbb{C}^*)^n$: the group $\mathcal{D}$ of invertible diagonal $n \times n$ matrices.
- F_n is a $\mathcal{D}$-invariant subvariety-*moment map*.
- The image of this map is the permutohedron P_n.





- The Associahedron
- The Catalan Number

Generealization:Newton-Okounkov body

- The Semigroup $S = \langle \mathbf{a} \rangle$ generated by $\mathbf{a} = \{a_1, \dots, a_n\}$ of positive integers.
- When $(a_1, \dots, a_n) = 1$, the semigroup $\langle \mathbf{a} \rangle$ is called a *numerical semigroup*
- Consider the k-algebra homomorphism $\phi_{\mathbf{a}} : K[x_1, \dots, x_n] \to k[t]$ given by $\phi_{\mathbf{a}}(x_i) = t^{a_i}$.
- The image of this map $\phi_{\mathbf{a}}$ is the semigroup ring $k[\mathbf{a}]$
- $k[\mathbf{a}] = K[x_1, \dots, x_n]/I_{\mathbf{a}}$ where $\ker \phi_{\mathbf{a}} = I(\mathbf{a})$.
- The $I_{\mathbf{a}}$ is the toric ideal of $k[\mathbf{a}]$.
- Since $(a_1, \dots, a_n) = 1$, there exists a smallest integer $r_i > 0$ such that $r_i a_i = \sum_{j \neq i} r_{ij} a_j$ for all $i = 1, \dots, n$.
- The $n \times n$ matrix $D(\mathbf{a}) := (r_{ij})$ where $r_{ii} := -r_i$ is called a principal matrix associated to $\mathbf{a}$.
- there is a number f such that $x > f \to x \in S$, a numerical semigroup. This number f is called the Frobenius number.
- f is the largest positive integer not in S.
- The semigroup S is symmetric if for all $x < f$, $x \in S$ if and only if $f - x \notin S$.

Clans, sects, and symmetric space closure orders

(joint with Mahir Can & Özlem Uğurlu)

Aram Bingham

T. U. Louisiana

ICERM Workshop
Geometry and Combinatorics from Root Systems
March 23, 2021

Symmetric Spaces

G, a connected reductive complex linear algebraic group (subgroup of $GL_n(\mathbb{C})$).

Definition

If θ is an automorphism of G of order 2 with $L = G^\theta$ the fixed-point subgroup, then we call G/L a symmetric space, and L a symmetric subgroup.

Theorem (Matsuki, '79)

A Borel subgroup of G acts on a symmetric space G/L with finitely many orbits.
(Symmetric spaces are spherical varieties.)

For simple G, classification corresponds to classification of real forms of simple Lie algebras (Cartan).

Those of Hermitian type come in four infinite families of pairs (G, L):

$$\begin{tikzcd} \text{① Type } AIII\text{: } (SL_{p+q}, S(GL_p \times GL_q)) & \text{② Type } CI\text{: } (Sp_{2n}, GL_n) \arrow[l, hook'] \arrow[l, phantom, shift right=4ex, "\textit{embeddings}"] \\ \text{③ Type } DIII\text{: } (SO_{2n}, GL_n) \arrow[u, hook'] & \text{④ Type } BDI\text{: } (SO_n, SO_2 \times SO_{n-2}) \arrow[ul, hook'] \end{tikzcd}$$

*** In these cases, $P = L \ltimes R_u(P)$ for P parabolic $\rightsquigarrow G/P$ is a Grassmannian!***

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1. Type $AIII$: $(SL_{p+q}, S(GL_p \times GL_q))$ $\longleftarrow$ 2. Type CI: (Sp_{2n}, GL_n)

embeddings

3. Type $DIII$: (SO_{2n}, GL_n)
4. Type BDI: $(SO_n, SO_2 \times SO_{n-2})$

*** In these cases, $P = L \ltimes R_u(P)$ for P parabolic $\rightsquigarrow$ G/P is a Grassmannian!***

Inclusion

Type	Symmetric Pair	B-orbits parametrized by	G/P
$AIII$	$(SL_{p+q}, S(GL_p \times GL_q))$	(p,q)-clans	$\mathrm{Gr}_p(\mathbb{C}^{p+q})$
CI	(Sp_{2n}, GL_n)	skew-symmetric (n,n)-clans	$\Lambda(n)$
$DIII$	(SO_{2n}, GL_n)	$DIII$ (n,n)-clans	$\mathrm{OGr}_n(\mathbb{C}^{2n})$

$$\{DIII\ (n,n)\text{-clans}\} \hookrightarrow \{\text{skew-symmetric } (n,n)\text{-clans}\} \hookrightarrow \{(n,n)\text{-clans}\}$$

Definition (Matsuki-Oshima '90, Yamamoto '97)

An (n,n)-clan is a string of $2n$ symbols, which are either $+$, $-$, or a natural number, such that:

1. If a number appears in the string then it must appear exactly twice.
2. There are the same number of $+$ and $-$ symbols.

Example/clarification: We consider $+1212-$ the same $(3,3)$-clan as $+2121-$.
Question: When is a symmetric space closure order equal to the **restriction of the type $AIII$ closure order?**

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Hermitian-type symmetric space closure order

We have two natural projection maps

$$G/P \xleftarrow{\pi} G/L \xrightarrow{\pi^-} G/P^-$$

(P^- is "opposite" parabolic, $L = P \cap P^-$.)

Bruhat order on clans is determined by (after Wyser '16, Gandini-Maffei '17):

1. images of orbits in G/P and in G/P^- ("sects"),
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Example: Clan $\gamma = +1212-$ has underlying involution $\sigma_\gamma = (2\ 4)(3\ 5)$.

Combinatorial gadgets

1. Compare images in G/P and G/P^- by containment of lattice paths.
2. Involutions are compared using rank control matrices (types A and C).

Example: $\sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xmapsto{R(\sigma)} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$, $\rho = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xmapsto{R(\rho)} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$,

so in this case $\sigma \leq \rho$ because each entry of $R(\sigma)$ is $\leq$ that of $R(\rho)$.

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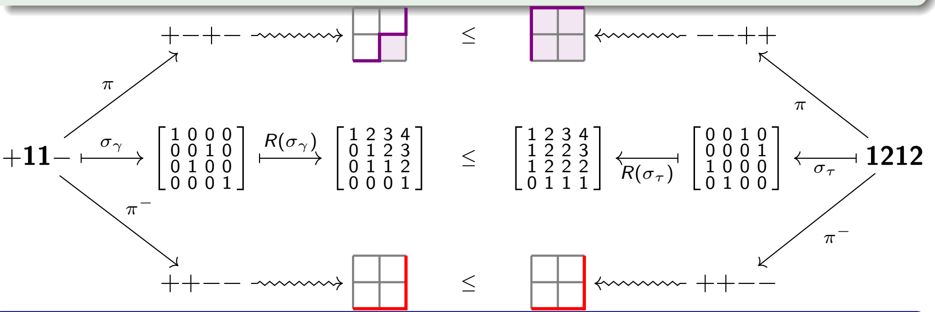
Example and result

For clans, $\gamma \leq \tau$ if and only if...

1. $\pi(\gamma) \leq \pi(\tau)$ in G/P (associated lattice path lies weakly below),
2. $\pi^-(\gamma) \leq \pi^-(\tau)$ in G/P^- (associated lattice path lies weakly above),
3. for involutions, $\sigma_\gamma \leq \sigma_\tau$.



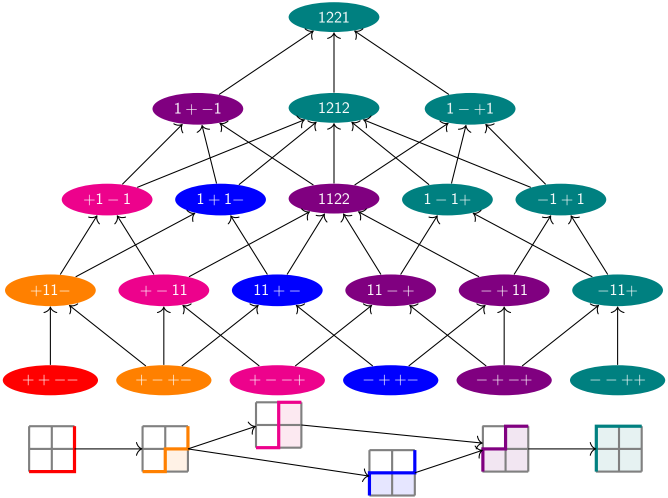
Theorem, (B., '20)

The Bruhat order on the type *CI* symmetric space is the restriction of the Bruhat order on the type *AIII* symmetric space to the skew-symmetric clans.

Remark: *DIII* fails to restrict at the Weyl group level, comparing involutions.

Thank You!

Inclusion order on Borel orbit closures in $SL_4/L_{2,2}$.



Bruhat order on Schubert cells in $Gr(2, 4)$.

A generating function for counting mutually annihilating matrices over a finite field

Yifeng Huang

University of Michigan

Mar. 23, 2021

Main Result

Theorem (H.) Let $\mathbb{F}_q$ be the finite field of q elements, $\mathrm{Mat}_n(\mathbb{F}_q)$ denote the set of $n \times n$ matrices over $\mathbb{F}_q$, and $\mathrm{GL}_n(\mathbb{F}_q)$ the set of invertible matrices therein. Then

$$\sum_{n=0}^{\infty} \frac{|\{A, B \in \mathrm{Mat}_n(\mathbb{F}_q) : AB = BA = 0\}|}{|\mathrm{GL}_n(\mathbb{F}_q)|} z^n$$
$$= \big((1-z)(1-q^{-1}z)(1-q^{-2}z)\dots\big)^{-2} H_q(z),$$

where $H_q(z)$ is a power series in z with infinite radius of convergence.

Techniques of the proof: Counting is easy. The factorization uses standard q-series identities involving Young diagrams and Durfee squares. $H_q(z)$ can be written down explicitly.

Geometric picture

Such generating functions correspond to affine $\mathbb{F}_q$-varieties in a systematic way:

$$\sum_{n=0}^{\infty} \frac{|\{A, B \in \mathrm{Mat}_n(\mathbb{F}_q) : AB = BA = 0\}|}{|\mathrm{GL}_n(\mathbb{F}_q)|} z^n \rightsquigarrow \{(x,y) : xy = 0\}$$

$$\left((1-z)(1-q^{-1}z)(1-q^{-2}z)\ldots\right)^{-1} = \sum_{n=0}^{\infty} \frac{|\mathrm{Mat}_n(\mathbb{F}_q)|}{|\mathrm{GL}_n(\mathbb{F}_q)|} z^n \rightsquigarrow \mathbb{A}^1_{\mathbb{F}_q}$$

If we denote the generating function associated to a variety X by $\widehat{Z}_X(z)$, then the main result can be restated as

$$\frac{\widehat{Z}_{\{xy=0\}}(z)}{\widehat{Z}_{\text{(two lines)}}(z)} \text{ is an entire function.}$$

Conjecture

Conjecture (informal)

The fact that $\{$two disjoint lines$\}$ is a resolution of singularity of $\{xy = 0\}$ is the geometric reason behind the main result.

Conjecture (formal)

Let X be any curve over $\mathbb{F}_q$ with only planar singularities, and assume $\widetilde{X}$ is a resolution of singularity of X. Then $\dfrac{\widehat{Z}_X(z)}{\widehat{Z}_{\widetilde{X}}(z)}$ is entire in z.

We remark that the question only depends on the type of the singularity. The main result implies the conjecture holds for nodes.

This phenomenon is seen for the generating function of Hilbert schemes (Göttsche–Shende '14, Refined curve counting on complex surfaces).

Other open questions

Even in the main result, the holomorphic factor $H_q(z)$ is explicit, its behavior is still mysterious.

- Can it be further factorized? (Most likely no, but maybe there is a natural weaker question to ask.)
- Does it have an “almost” functional equation?

Here are the observed data for the zeros of $H_q(z)$:

- There seem to be infinitely many zeros, namely, $z_1, z_2, \ldots$ in first quadrant, and their complex conjugates.
- $z_{n+1} \approx q z_n$, and $|z_n| \approx q^{n-\frac{1}{2}}$. For $q = 4$,

$$\begin{aligned}
z_1 &= 0.41614 + 1.72467i, & |z_1| &= 1.77288 < 2 = q^{1/2} \\
z_2 &= 1.65483 + 7.60611i, & |z_2| &= 7.78405 < 8 = q^{3/2} \\
z_3 &= 6.62192 + 31.08907i, & |z_3| &= 31.7865 < 32 = q^{5/2} \\
z_4 &= 26.4883 + 125.0116i, & |z_4| &= 127.787 < 128 = q^{7/2}
\end{aligned}$$

Thank you!