Geometry of semi-infinite flag manifolds

Longer version

Syu Kato

Department of Mathematics, Kyoto University

March 23, 2021

Geometry and Combinatorics from Root Systems @ ICERM

Main reference: arXiv:1810.07106
Application to enumerative problems:
another seminar on April 1st 3:30–4:30pm at ICERM

Recollection on flag manifolds

G : semi-simple, simply connected algebraic group over $\mathbb{C}$
$T \subset B$: maximal torus of $G \subset$ maximal solvable subgroup of G
$B \subset P \subset G$: a (standard) parabolic subgroup of G

$$X_{G,P} := G/P : \text{(partial) flag manifold.}$$

It carries information about

- Representations of semi-simple Lie algebras (Borel, Beilinson-Bernstein, ...);
- (generalized) Littlewood-Richardson rule (Lascoux, ...);
- Classifying spaces of classical groups (Fulton, ...).

Geometry of affine flag manifolds

There are several reasons to consider the *affine* version of the flag manifolds:

- Representations of affine Lie algebras;
- Representations of p-adic groups;
- Quantum cohomology of flag manifolds ($\Leftarrow$ LR coefficients);
- Wess-Zumino-Witten model in conformal field theory ($\Leftarrow$ BGL);

Affine version means we replace G with $G(\!(z)\!)$ (the formal loop group of G), defined as the set of $\mathbb{C}(\!(z)\!)$-valued points $G(\mathbb{C}(\!(z)\!))$ of G.

Axiom of choice yields $\mathbb{C}(\!(z)\!) \subset \mathbb{C}$, thus we have $G(\!(z)\!) \subset G$.

If we take account into the topology of $\mathbb{C}(\!(z)\!)$, then things gets complicated:

- $\mathbb{C}[\![z]\!]$ is a (pro-)algebraic group;
- $\mathbb{C}(\!(z)\!)$ is not a (pro-)algebraic group.

$\Leftarrow$ a countably infinite dimensional vector space is not a dual of a vector space.

Affine flag manifolds

Consider the pro-algebraic group $G[\![z]\!] := G(\mathbb{C}[\![z]\!])$, consider the evaluation at 0

$$\mathtt{ev} : G[\![z]\!] \longrightarrow G$$

and define the affine flag manifolds to be

$$X^{\mathrm{af}}_{G,P} := G(\!(z)\!)/\mathtt{ev}^{-1}(P).$$

The case $P = G$ is the affine Grassmannian. They yield satisfactory output in many applications, but we may consider the induction of a $B(\!(z)\!)$-representations to $G(\!(z)\!)$. In that case, it is a serious theorem that $X^{\mathrm{af}}_{G,P}$ gives a correct output.

The goal of this talk

is to exhibit a basic material on the geometry of $G(\!(z)\!)/B(\!(z)\!)$ and explain they share some similar aspect with $X^{\mathrm{af}}_{G,P}$ but the underlying geometry is different.

Semi-infinite flag manifolds

Since $G(\!(z)\!)/B(\!(z)\!)$ is just the scalar restriction of G/B through $\mathbb{C}(\!(z)\!) \subset \mathbb{C}$, it is not interesting enough. In the late 1970s to early 1980s, Lusztig and Drinfeld noticed that more better version would be

$$G(\!(z)\!)/\left(T[\![z]\!] \cdot N(\!(z)\!)\right), \qquad \text{where } N = [B, B].$$

Finkelberg and Mirković noticed that this set cannot define a separated ($\doteq$ Hausdorff) algebraic variety, and modified the definition as

$$\mathbf{Q}^{\mathrm{rat}} := G(\!(z)\!)/\left(T \cdot N(\!(z)\!)\right).$$

This is what we call the formal model of (full) semi-infinite flag manifolds.

Plücker embeddings of semi-infinite flag manifolds

At this moment, we have $\mathbf{Q}^{\mathrm{rat}} := G((z))/\left(T \cdot N((z))\right)$ as a set. We embed it into projective spaces to equip a scheme structure: Find an irreducible G-module V with (a unique) B-eigenvector v. We form

$$\mathbb{P}(V((z))) := \left(V \otimes_{\mathbb{C}} \mathbb{C}((z)) \setminus \{0\}\right)/\mathbb{C}^{\times} \supset \mathbb{P}(V[\![z]\!]z^{-n}) := \left(V \otimes_{\mathbb{C}} \mathbb{C}[\![z]\!]z^{-n} \setminus \{0\}\right)/\mathbb{C}^{\times}$$

Then, $N((z))$ fixes $v \otimes 1$, and induces a map

$$\phi_V : \mathbf{Q}^{\mathrm{rat}} = G((z))/\left(T \cdot N((z))\right) \longrightarrow \mathbb{P}(V((z))).$$

We still need to justify $\mathbb{P}(V((z)))$. We have

$$\mathbb{P}(V[\![z]\!]z^{n}) \cong \mathrm{Proj}\, S^{\bullet}\left(\bigoplus_{j \le n} V^{*} \otimes \xi^{n}\right) \qquad \xi^{-1} \leftrightarrow z \text{ is dual}$$

Thus, $\mathbb{P}(V((z))) = \bigcup_n \mathbb{P}(V[\![z]\!]z^{-n})$ is an ind-scheme.

Ind-scheme structure of $\mathbf{Q}^{\mathrm{rat}}$

Lemma (See p12 for standard example of $\mathbb{V}$)

There are $(\mathrm{rk}\, G)$ collections $\mathbb{V}$ of irreducible G-modules such that the product map $\phi_{\mathbb{V}} := \prod_{V \in \mathbb{V}} \phi_V : \mathbf{Q}^{\mathrm{rat}} \longrightarrow \prod_{V \in \mathbb{V}} \mathbb{P}(V(\!(z)\!))$ is an inclusion of sets. Moreover, the image is Zariski closed when restricted to each $\prod_V \mathbb{P}(V[\![z]\!]z^{-n_V})$.

Corollary

The embedding $\phi_{\mathbb{V}}$ defines an ind-scheme structure $\mathbf{Q}^{\mathrm{rat}}_{\mathbb{V}}$ on $\mathbf{Q}^{\mathrm{rat}}$.

Theorem

There exists a "universal" ind-scheme that maps to the scheme structure on $\mathbf{Q}^{\mathrm{rat}}$ defined by every possible $\phi_{\mathbb{V}}$.

So, we identify $\mathbf{Q}^{\mathrm{rat}}$ with this universal ind-scheme in the below.

Warning

Unlike the case of $X_{G,B}$, the scheme structure $\mathbf{Q}^{\mathrm{rat}}_{\mathbb{V}}$ heavily depends on $\mathbb{V}$! In fact, the map $\mathbf{Q}^{\mathrm{rat}} \to \mathbf{Q}^{\mathrm{rat}}_{\mathbb{V}}$ is not birational in general (even for $G = SL(2)$).

What is written in this slide needs correction. See the last slide.

Structural results on $\mathbf{Q}^{\mathrm{rat}}$

We set $\mathbf{I} := \mathrm{ev}^{-1}(B) \subset G[\![z]\!]$ (the Iwahori subgroup).

Theorem (Iwahori-Matsumoto, Lusztig)

1. Set of $\mathbf{I}$-orbits of $\mathbf{Q}^{\mathrm{rat}}$ is in bijection with W_{af}, the affine Weyl group of G;
2. Each $\mathbf{I}$-orbit of $\mathbf{Q}^{\mathrm{rat}}$ is infinite-dimensional and is infinite-codimensional. It carries a unique $(T \times \mathbb{G}_m)$-fixed point p, where $\mathbb{G}_m$ acts (only) on z;
3. The closure relation between the $\mathbf{I}$-orbits of $\mathbf{Q}^{\mathrm{rat}}$ is the inverse generic/semi-infinite order on W_{af} (denoted by $\leq_{\frac{\infty}{2}}$).

Example $G = SL(2)$

$W_{\mathrm{af}} = \mathfrak{S}_2 \ltimes \mathbb{Z}\alpha^\vee$, where $\mathfrak{S}_2 = \{1, s\}$ and $\alpha^\vee$ is the simple coroot. We have

$$\mathbf{Q}^{\mathrm{rat}} \cong \mathbb{P}(\mathbb{C}^2(\!(z)\!)) \quad \text{and} \quad p_{(s,m\alpha^\vee)} = [e_1 \otimes z^m], p_{(e,m\alpha^\vee)} = [e_{-1} \otimes z^m].$$

Richardson varieties of $\mathbf{Q}^{\mathrm{rat}}$

For each $w \in W_{\mathrm{af}}$, we set $\mathbf{Q}(w)$ the Zariski closure of the corresponding $\mathbf{I}$-orbit.

Assume $V \cong V^*$ as a G-module (or $(\mathbb{V})^* = \mathbb{V}$ as a set). Consider an embedding

$$\mathbb{P}(V \otimes \mathbb{C}((z))) \subset \mathbb{P}(V \otimes \mathbb{C}[\![z, z^{-1}]\!]) := (V \otimes \mathbb{C}[\![z, z^{-1}]\!] \setminus \{0\})/\mathbb{C}^\times$$

The RHS $\mathbb{P}(V \otimes \mathbb{C}[\![z, z^{-1}]\!])$ is in fact a scheme with symmetry θ defined as

$$\theta : v \otimes z^m \mapsto v^* \otimes z^{-m} \qquad v \in V, m \in \mathbb{Z}.$$

Lemma

The map θ sends the $(T \times \mathbb{G}_m)$-fixed point of $\mathbf{Q}^{\mathrm{rat}}$ corresponding to $w \in W_{\mathrm{af}}$ to ww_0, where w_0 is the longest element of the finite Weyl group W of G.

Definition

For each $w, w' \in W_{\mathrm{af}}$, we define the Richardson variety of $\mathbf{Q}^{\mathrm{rat}}$ as:

$$\mathcal{Q}(w, w') := \mathbf{Q}(w) \cap \theta(\mathbf{Q}(w'w_0)).$$

An interpretation of Richardson varieties of $\mathbf{Q}^{\mathrm{rat}}$

Let $Q_+^\vee \subset Q^\vee$ be the (positive) coroot span. We have an embedding $Q^\vee \subset W_{\mathrm{af}}$ as the translation part, and $Q_+^\vee \subset H_2(X_{G,B}, \mathbb{Z})$ is identified with the classes represented by algebraic curves.

Theorem

Let $\beta \in Q_+^\vee$. Then, $\mathcal{Q}(0, \beta)$ is isomorphic to the Drinfeld ($\doteq$ the smallest nice) compactification (a.k.a quasi-map spaces) of the space of maps

$$f : \mathbb{P}^1 \longrightarrow X_{G,B}$$

with $f_*[\mathbb{P}^1] = \beta \in H_2(X_{G,B}, \mathbb{Z})$. Some interpretation exists for every $w, w' \in W_{\mathrm{af}}$, and $\mathcal{Q}(w, w')$ is always normal.

Being subschemes of $\mathbf{Q}(e)$, we have

$$\mathcal{Q} := \bigcup_{\beta \in Q_+^\vee} \mathcal{Q}(0, \beta) \subset \mathbf{Q}(e).$$

This is a Zariski dense subset, and we refer this to be the ind-model of $\mathbf{Q}(e)$.

Comment on the structure of $\mathbf{Q}^{\mathrm{rat}}$

Thanks to the moduli interpretation, we have the following result:

Theorem (Braverman-Finkelberg)

The singularity of $\mathcal{Q}(0,\beta)$, that is the same as the structure of the singularity of the Zastava space, has an approximation by the affine Grassmannian slices of G.

Someone is claiming that the geometry of $\mathcal{Q}(0,\beta)$ (or $\mathcal{Q},\mathbf{Q}$) is the "same" as that of affine Grassmannian of G based on this result. However, let us stress that

- $\mathcal{Q}(0,\beta)$ (or $\mathcal{Q},\mathbf{Q}$) are just projective spaces for $G = SL(2)$;
- the affine Grassmannian of $SL(2)$ is already highly singular.

Thus, all the results that claim equivalences of some structures between $\mathbf{Q}$ and affine Grassmannian in fact contains the assertion that such a structure do not detect some difference of singularities.

Lesson

Even if the final result looks as expected, funny things tend to happen in the middle when working on $\mathbf{Q}^{\mathrm{rat}}$. We will see another instance in the last.

Compatible projective embeddings

Let $\varpi_1, \ldots, \varpi_r$ be the collection of fundamental weights of G ($\mathrm{rk}\, G = r$). Let $V(\varpi_i)$ be the corresponding finite-dimensional representation of G. We can take $\mathbb{V} = \{V(\varpi_i)\}_{1 \le i \le r}$ and have embeddings

$$\mathcal{Q}(0,\beta) \subset \mathcal{Q}(0,\beta+\beta') \subset \mathbf{Q}(e) \subset \mathbf{Q}^{\mathrm{rat}} \subset \prod_{i=1}^{r} \mathbb{P}(V(\varpi_i)(\!(z)\!)) \qquad \beta' \in Q_+^\vee$$

Thus, the i-th embedding induces a line bundle $\mathcal{O}(\varpi_i)$ on each of them and the their tensor products yields a line bundle $\mathcal{O}(\lambda)$ for every weight λ in the integral weight lattice P of G.

Theorem (K)

If $\langle \beta, \lambda \rangle \ge 0$ for every $\beta \in Q_+^\vee$, we have

$$H^{>0}(\mathcal{Q}(0,\beta), \mathcal{O}(\lambda)) = 0$$

and we have a surjection

$$H^0(\mathcal{Q}(0,\beta+\beta'), \mathcal{O}(\lambda)) \longrightarrow\!\!\!\!\!\rightarrow H^0(\mathcal{Q}(0,\beta), \mathcal{O}(\lambda)) \qquad \beta' \in Q_+^\vee.$$

Borel-Weil-Bott theorem for Q

Theorem (K-Naito-Sagaki, K)

We have

$$H^i(\mathbf{Q}(e), \mathcal{O}(\lambda)) \cong \begin{cases} \mathbb{W}(\lambda)^\vee & (i = 0, \lambda \in P_+) \\ \{0\} & (else) \end{cases},$$

where $\mathbb{W}(\lambda)$ is the global Weyl module of $\mathrm{Lie}\, G \otimes \mathbb{C}[z]$ in the sense of Chari-Pressley. If $\langle \beta, \lambda \rangle \geq 0$ for every $\beta \in Q_+^\vee$, the natural map

$$H^0(\mathbf{Q}(e), \mathcal{O}(\lambda)) \to \varprojlim_\beta H^0(\mathcal{Q}(0, \beta), \mathcal{O}(\lambda))$$

is a dense embedding with respect to the topology of the latter. Similar results holds for every $\mathbf{Q}(w)$ ($w \in W_{\mathrm{af}}$).

Remark

We also have a version of the Demazure character formula in this setting, that does not compute the character, but characterize them by difference equations.

Idea on equivariant K-group for $\mathbf{Q}^{\mathrm{rat}}$

Remark

The scheme $\mathbf{Q}(e)$ is far from Noetherian, and hence defining $K(\mathbf{Q}(e))$ in a usual manner have serious difficulties.

Observation

The projective coordinate ring of $\mathbf{Q}(e)$ is "graded artin" if we incorporate degrees arising from $(T \times \mathbb{G}_m)$-action.

By the numerical counter-parts of the Serre description of the coherent sheaves on projective varieties, we have

$$K_T(X_{G,B}) \hookrightarrow \{f : P \to \mathbb{Z}[P]\}/\{f \text{ s.t. } f(\lambda) \equiv 0 \text{ for } \lambda \gg 0\}$$

by

$$[\mathcal{F}] \mapsto \left(\lambda \mapsto \mathrm{gch}\, H^0(\mathcal{F} \otimes_{\mathcal{O}_{X_{G,B}}} \mathcal{O}(\lambda))\right).$$

Equivariant K-group for $\mathbf{Q}^{\mathrm{rat}}$

Theorem (K-Naito-Sagaki)

There exists a subspace $K_{T\times\mathbb{G}_m}(\mathbf{Q}^{\mathrm{rat}})$ of

$$\{f : P \to \mathbb{Z}(\!(q^{-1})\!)[P]\}/\{f \text{ s.t. } f(\lambda) \equiv 0 \text{ for } \lambda \gg 0\}$$

that contains the functionals (that we call $[\mathcal{O}_{\mathbf{Q}(w)}]$)

$$P \ni \lambda \mapsto \mathrm{gch}\, H^0(\mathbf{Q}(w), \mathcal{O}_{\mathbf{Q}(w)}(\lambda)) = \chi_{T\times\mathbb{G}_m}(\mathcal{O}_{\mathbf{Q}(w)}(\lambda)) \quad \forall w \in W_{\mathrm{af}}$$

as $\mathbb{Z}[q^{\pm}][T]$-topological basis. It is closed under (equivariant) line bundle twists.

Corollary

For each $w \in W_{\mathrm{af}}, \lambda \in P$, we have a formula of the form

$$[\mathcal{O}_{\mathbf{Q}(w)}(\lambda)] = \sum_{v \leq_{\frac{\infty}{2}} w} a_w^v(\lambda)[\mathcal{O}_{\mathbf{Q}(v)}] \qquad a_w^v(\lambda) \in \mathbb{Z}[q^{\pm}][T].$$

If $\lambda \in P_+$, then $a_w^v(\lambda) \in \mathbb{Z}_{\geq 0}[q^{\pm}][T]$. But the RHS can be an infinite sum.

Global sections on Richardson varieties of $\mathbf{Q}^{\mathrm{rat}}$

We set $P_+ := \{\lambda \in P \mid \langle \beta, \lambda \rangle \geq 0$ for every $\beta \in Q_+^\vee\}$. For $\lambda \in P_+$, let $\mathcal{O}(\lambda)$ to be the pullback of $\mathcal{O}(1)$ through $\phi_{V(\lambda)}$, and extend to P by $\otimes$.

Theorem

For $w \in W_{\mathrm{af}}$ and $\lambda \in P_+$, we have $a_w^v(\lambda) \in \mathbb{Z}_{\geq 0}[q^\pm][T]$ ($v \in W_{\mathrm{af}}$) such that

$$\chi_{T \times \mathbb{G}_m}(\mathcal{Q}(w, w'), \mathcal{O}(\lambda)) = \mathrm{gch}\, H^0(\mathcal{Q}(w, w'), \mathcal{O}(\lambda)) = \sum_{w' \leq_{\frac{\infty}{2}} v \leq_{\frac{\infty}{2}} w} a_w^v(\lambda)$$

for every $w' \in W_{\mathrm{af}}$. In addition, we have $H^{>0} \equiv 0$ in this case.

The proof of theorems requires Frobenius splitting, propagated as:

$$\text{thin affine} \rightsquigarrow \text{thick affine} \rightsquigarrow \mathbf{Q} \rightsquigarrow \mathcal{Q}$$

Remark

Unlike the case of $X_{G,B}$, the open part of $\mathcal{Q}(w, w')$ (i.e. complement of the union of the smaller ones) is (highly) singular.

Global sections on Richardson varieties of $\mathbf{Q}^{\mathrm{rat}}$

We set $P_+ := \{\lambda \in P \mid \langle \beta, \lambda \rangle \geq 0 \text{ for every } \beta \in Q_+^\vee\}$. For $\lambda \in P_+$, let $\mathcal{O}(\lambda)$ to be the pullback of $\mathcal{O}(1)$ through $\phi_{V(\lambda)}$, and extend to P by $\otimes$.

Theorem

For $w \in W_{\mathrm{af}}$ and $\lambda \in P_+$, we have $a_w^v(\lambda) \in \mathbb{Z}_{\geq 0}[q^{\pm}][T]$ ($v \in W_{\mathrm{af}}$) such that

$$\chi_{T \times \mathbb{G}_m}(\mathcal{Q}(w, w'), \mathcal{O}(\lambda)) = \mathrm{gch}\, H^0(\mathcal{Q}(w, w'), \mathcal{O}(\lambda)) = \sum_{w' \leq_{\frac{\infty}{2}} v \leq_{\frac{\infty}{2}} w} a_w^v(\lambda)$$

for every $w' \in W_{\mathrm{af}}$. In addition, we have $H^{>0} \equiv 0$ in this case.

Remark

Unlike the case of $X_{G,B}$, the open part of $\mathcal{Q}(w, w')$ (i.e. complement of the union of the smaller ones) is (highly) singular.

Remark

Above is surprising as Richardson's proof is very general transversality argument. His logic cannot be applied here as it requires $G(\mathbb{C}[\![z, z^{-1}]\!])$ to be a Lie group, but the object $G(\mathbb{C}[\![z, z^{-1}]\!])$ simply does not make sense even set-theoretically

Ind-scheme structure of $\mathbf{Q}^{\mathrm{rat}}$ – correction

Lemma (This is not true in general. Standard example in p12 is OK)

There are $(\mathrm{rk}\, G)$ collections $\mathbb{V}$ of irreducible G-modules such that the product map $\phi_{\mathbb{V}} := \prod_{V \in \mathbb{V}} \phi_V : \mathbf{Q}^{\mathrm{rat}} \longrightarrow \prod_{V \in \mathbb{V}} \mathbb{P}(V(\!(z)\!))$ is an inclusion of sets. Moreover, the image is Zariski closed when restricted to each $\prod_V \mathbb{P}(V[\![z]\!]z^{-n_V})$.

Theorem (this is the original theorem as in arXiv:1810.07106)

There exists a "universal" ind-scheme that maps to the any ind-scheme structure on $\mathbf{Q}^{\mathrm{rat}}$ that is compatible with the $\mathbf{I}$-orbits and its closure relations.

To obtain ind-scheme structure on $\mathbf{Q}^{\mathrm{rat}}$ (or $\mathbf{Q}(e)$) that is distinct from Theorem is to take the jet-scheme of the basic affine space $\overline{G/N}$, defined as

$$\overline{G/N} := \mathrm{Spec}\, \mathbb{C}[G]^N.$$

Its ∞-jet scheme $(\overline{G/N})_\infty$, has the right T-action with open dense subset $(\overline{G/N})^f_\infty$ on which T-action is free. We have

$$\mathbf{Q}(e) = (\overline{G/N})^f_\infty / T$$

as sets but not as schemes. I apologize this mistake itself, as well as try to add something too premature (not in my papers) in an ad hoc manner.