Q'm Geom Satake at r.o.u. and

(Schubert calculus for) $G(m,m,n)$

Ben Elias

UOregon, currently IAS

Featuring joint work with Ben Young + Daniel Juteau

$$\begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & -q & 0 & 0 & -q^{-1} \\ -q^{-1} & 2 & -q & 0 & 0 \\ 0 & -q^{-1} & 2 & -q & 0 \\ 0 & 0 & -q^{-1} & 2 & -q \\ -q & 0 & 0 & -q^{-1} & 2 \end{pmatrix}$$

Let (W,S) be the Coxeter system associated to a Weyl group

$W \curvearrowright V$ the reflection repn w/ basis $\{\alpha_s\}_{s\in S}$. Action is determined by the Cartan matrix (a_{st}) via

$$s(\alpha_t) = \alpha_t - a_{st}\alpha_s.$$

Let $R = \mathrm{Sym}^*(V)$ graded w/ $\deg V = 2$. $W \curvearrowright R$. R^W the invariant subring.

Geometry: $R = H^*_T(pt) = H^*_B(pt)$ $\qquad R^W = H^*_G(pt)$

$$C := R/(R^W_+) = H^*(G/B)$$

coinvariant ring

Thm: (Shephard-Todd, Chevalley): 1) R^W is a poly ring w/ same # of generators as R.

2) R is graded free over R^W of finite rank $= \# W$

Thm: (Demazure, "Upgraded Chevalley Thm") $R^W \subset R$ is a (graded) Frobenius extension!

I.e., $\mathrm{Ind} \vdash \mathrm{Res} \vdash \mathrm{Ind}$ (up to shift). Equivalently, $\exists$ R^W-linear map $\partial_W : R \to R^W$

s.t. $(f,g) \mapsto \partial_W(fg)$ has dual bases.

Frobenius trace

Explicitly,

$$\partial_W(f) = \frac{\sum_{w \in W} (-1)^{\ell(w)} w(f)}{\prod_{\beta \in \Phi^+} \beta}$$

← project to sign rep of W

← generates R^{-W} as R^W-module

Geometry: $H^*_B(pt) \xrightarrow{\pi_*} H^*_G(pt)$ is integration along G/B fibers, "Poincaré Duality"

$\Rightarrow H^*(G/B)$ is Frobenius alg. over $H^*(pt)$ ← ignoring subtleties with choice of base ring

Demazure's proof that ∂_W is frob. trace really used:

__Alg / Comb.__ ∂_w can also be constructed using the NilCoxeter algebra.

For $s \in S$, let $\partial_s(f) := \dfrac{f - sf}{\alpha_s}$ ⟵ this is $\partial_{w_{\langle s \rangle}}$ for parabolic subgp $\langle s \rangle$

Inside $\mathrm{End}_{R^W}(R) \cong \mathrm{Mat}_{\#W}(R^W)$ these generate NC_W.

Presentation: $NC_W = \langle \partial_s \mid \partial_s^2 = 0,\ \text{braid relns} \rangle$ $\Longrightarrow$ $\partial_w := \partial_{s_1} \cdots \partial_{s_d}$ for red exp

__Thm:__ $\partial_W = \partial_{w_0}$. Any ∂_w can be "extended" to $\partial_{w_0} = \partial_v \partial_w$ by a unique v. ⟵ used in D's proof

---

__Use 1:__ Relating Frob ext for different parabolics $I \subset S$

$$\begin{tikzcd} R^W \arrow[r, hook] & R^I \arrow[r, hook] \arrow[l, bend left, "\partial_{w_0 w_I^{-1}}"] & R \arrow[l, bend left, "\partial_{w_I}"] \arrow[ll, bend right, "\partial_{w_0}"'] \end{tikzcd}$$

__Use 2:__ Finding dual bases for R over R^W. "Schubert Calculus" ⟵ (don't hate me)

$\{\partial_w(f)\}$

Rmk: For G/B, not so bad. For R/R^W, harder...

Generalizations:

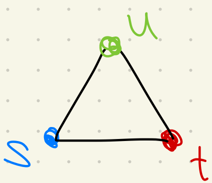
1) W finite Coxeter gp: Chevalley ✓ Upgraded ✓ NC ✓ Geometry ✗ Shadows of geometry ✓

E.g: C has Hard Lefschetz / Hodge-Riemann.

2) W (infinite) Coxeter gp: Chevalley ✗ (maybe R^W is poly ring but $R^W \subset R$ infinite extension)

But OK for $R^I \hookrightarrow R$ when $W_I \subset W$ is finite I is finitary.

Ex:



$$\begin{tikzcd}
 & R \arrow[dl, bend right, "\partial_s"'] \arrow[d] \arrow[dr, bend left] & \\
R^s \arrow[d] \arrow[dr] & R^t \arrow[dl] \arrow[dr] & R^u \arrow[dl] \arrow[d] \\
R^{st} & R^{su} & R^t \\
 & \cancel{R^{stu}} &
\end{tikzcd}$$

Commuting Frobenius cube

The input to the construction of singular Soergel bimodules

Alg + comb. category encoding geom. of $P_I \backslash G / P_J$

3) W a $\mathbb{C}$x Refl Gp: Chevalley ✓ Upgrade ?? NC ?? Geometry ✗ Shadows ??

(For $G(m,1,n)$ and $G(m,m,n)$, Shoji-Rampetas did some but different from our approach.)

4) Today q-deformation R_q for W in type $\tilde{A}_n$.

Where Geometric Satake relates Singular Soergel Bimodules for W_{aff} with Rep $\mathfrak{g}^L$,

Shadows of GS relates Singular Soergel Bimodules$_q$ with Rep $U_q(\mathfrak{sl}_n)$

and interesting stuff happens at q = roots of unity!

Ref: Quantum Satake in type A: part I (E. '14)

$n=5$

$\tilde{A}_4$

$$\begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & -q & 0 & 0 & -q^{-1} \\ -q^{-1} & 2 & -q & 0 & 0 \\ 0 & -q^{-1} & 2 & -q & 0 \\ 0 & 0 & -q^{-1} & 2 & -q \\ -q & 0 & 0 & -q^{-1} & 2 \end{pmatrix}$$

Where did that come from?

Orig., ad hoc.

Work in prep. w/ G. Williamson: new geom explain!!

$n=2$

$\tilde{A}_1$

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & -[2] \\ -[2] & 2 \end{pmatrix} \quad [2]=q+q^{-1}$$

Rmk: When $a_{st}a_{ts}=1$ then $m_{st}=3$ so still action of W_{aff} on $\mathbb{Z}[q,q^{-1}]\langle \alpha_s \rangle = V_q$

Rmk: Lusztig proved $\exists$ of 1-param family deforming V in types $\tilde{A}, \tilde{C}$.

But our parametrization is awesome, b/c GS_q and...

Recall: $W_{aff} \cong W_{fin} \ltimes \Lambda_{rt} = S_n \ltimes \mathbb{Z}^{n-1}$ $\longleftarrow$ elts in $\mathbb{Z}^n$ where $\sum$ coords $= 0$.

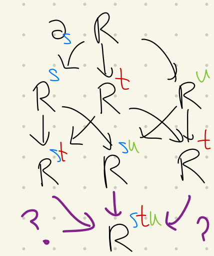
Thm: When $q = \zeta_{mn}$ is a primitive nm^{th} r.o.u. then

$$\begin{tikzcd} W_{aff} \arrow[r] \arrow[d, two heads] & \mathrm{End}(V_q) \\ W_{aff}/m\cdot\Lambda_{rt} \arrow[ur, hook] & \end{tikzcd}$$

$W_{aff}/m\cdot\Lambda_{rt} \cong G(m,m,n)$ but not your standard presentation!!!

So S-T-Chev says $R_q^{W_{aff}} \subset R_q$ is finite free!! rank $= n!\, m^{n-1}$

Upgrade? NC? Schubert?



Prop: $W_{aff}/m\cdot\Lambda_{rt}$ still has a sign repn. Orbit of α_s in $\mathbb{P}(V_q)$ is finite.

$\partial_w(f) := \sum (-1)^{\ell(w)} w(f) / (\prod \beta)$ still makes sense. Agrees w/ frob trace if one exists

Ex: $n=2$. $q=\zeta_{2m}$ $\begin{pmatrix} 2 & -2\cos\left(\frac{\pi}{m}\right) \\ -2\cos\left(\frac{\pi}{m}\right) & 2 \end{pmatrix}$ is the cartan matrix for the Dihedral Group $I_2(m)$!

It's a Coxeter gp, so have Upgraded ✓

Setting $q=\zeta_{2m}$ imposes new reln on NC_{aff}. Now $\underbrace{\partial_s \partial_t \cdots}_{m} = \underbrace{\partial_t \partial_s \cdots}_{m}$ becomes a f.d. algebra.

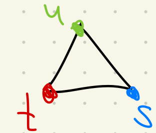
Aside: $n=2$ case explains general case. Let $\alpha_{long} = \alpha_1 + \ldots + \alpha_{n-1}$ highest finite root

t_{long} = longest finite reflection

Then $s_0 \cdot t_{long}$ = translation by $\alpha_{long} \in \Lambda_{rt}$.

Cartan for $\{\alpha_0, \alpha_{long}\}$ is $\begin{pmatrix} 2 & -[2] \\ -[2] & 2 \end{pmatrix}$ so $(s_0 t_{long})^m = 1$.

Ex: $n=3$, $m=2$ $G(2,2,3) \cong S_4$ w/ unusual presentation.



$s \mapsto (12)$
$t \mapsto (13)$
$u \mapsto (14)$

$W_{aff} / stsu = usts$

#W in each length: $1, 3, 6, 9, 5$ $\mathrm{gdim}_{R^W} R$: $1,3,5,6,5,3,1$

The operators $\partial_s, \partial_t, \partial_u$ generate a f.d. algebra NC_q w/ relns:

- $\partial_s^2 = \partial_t^2 = \partial_u^2 = 0$
- $\partial_t \partial_s \partial_t = q\, \partial_s \partial_t \partial_s$, etc.

true $\forall m$

- $\partial_s \partial_t \partial_u \partial_s + q^m \partial_t \partial_u \partial_s \partial_t + q^{2m} \partial_u \partial_s \partial_t \partial_u = 0$

  and flipped version

Thm: Generalizes to all m, reln in length $2m$.

- $\partial_t \partial_u \partial_s \partial_t \partial_u + \partial_u \partial_t \partial_s \partial_u \partial_t = \partial_t \partial_u \partial_s \partial_u \partial_t + \partial_u \partial_t \partial_s \partial_t \partial_u$

  and rotations

← Generalization: mystery!

Top degree is Frob trace!

$\mathrm{gdim}\, NC_q$

1
3
6
9
10
6
1

"longest element"

$\dim NC_q = 36$

$n=3$
gdim NC_q

<u>Note</u>: Poincaré of W_{aff}, NC_{aff} is 1, 3, 6, 9, 12, 15, …

$m=3$ example

Imposing two known relns in degree $2m$ makes it f.d., 1, 3, 6, 9, 12, 15, 16, 15, 12, 9, 6, 3, 1

deg = \ m =	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1
1	3	3	3	3	3	3	3
2	6	6	6	6	6	6	6
3	9	9	9	9	9	9	9
4	10	12	12	12	12	12	12
5	3 6	15	15	15	15	15	15
6	1	16	18	18	18	18	18
7		15	21	21	21	21	21
8		6	22	24	24	24	24
9	6	1	21	27	27	27	27
10			18	28	30	30	30
11		9	6	27	33	33	33
12			1	24	34	36	36
13				21	33	39	39
14				6	30	40	42
15			12	1	27	39	45
16					21	36	46
17				3	6	33	45
18					1	30	
19						21	
20					6	6	
21						1	
22							
DIM	36	84	153	243	351	480	630

deg =	12	13	14	15	16	17
32	48	60	72	84	94	96
33	45	57	69	81	93	99
34	21	54	66	78	90	100
35	21 6	51	63	75	87	99
36	1	48	60	72	84	96
37	24	21	57	69	81	93
38		6	54	66	78	90
39		1	50	63	75	87
40		1	21	60	72	84
41		21	6	57	69	81
42			1	52	66	78
43			2	21	63	75
44				6	60	72
45			18	1	52	69
46				5	21	66
47				3	6	63
48					1	52
49					8	21
50						6
51						1
DIM	1440	1694	1970	2266	2581	2917

It's utter chaos, but in all cases we've checked the word

$$\underbrace{\underbrace{stsu}_{t_{long}}\underbrace{u}_{s_0}\underbrace{stsu}_{t_{long}}\underbrace{u}_{s_0}stsu\ldots}_{3m}$$

is equal to ∂_w up to unit (and is Frob ext).

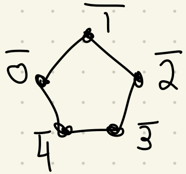
also, words obtained by symmetry

If true $\forall m$, we're close to proof of Frob. ext.

ANY IDEAS?

THANKS

Geometric Satake Primer (type A)

$\widetilde{A}_{n-1}$ 

vertices form a group $\Omega \cong \Lambda_{wt}/\Lambda_{rt}$ for $\mathfrak{sl}_n$

I think of $\mathrm{Rep}^{\Omega}\,\mathfrak{sl}_n$ as a 2-category. $\mathrm{Ob} = \Omega$. $\mathrm{Hom}(x, x') = \mathrm{Rep}_{x'-x}$

Think: If V has h.w. in $x'-x$ then $V \otimes (-)$ goes from Rep_x to $\mathrm{Rep}_{x'}$

SingSBim is a 2-cat. Ob: finitary subsets of $\widetilde{A}_{n-1}$

$$\mathrm{Hom}(I, J) \subset (R^J, R^I)\text{-bimodules}$$

generated by restriction ${}_{R^J}R^I \otimes_{R^I} (-)$ and induction ${}_{R^I}R^I \otimes_{R^J} (-)$ for $I \subset J$ $\Leftrightarrow R^J \subset R^I$

AND DIRECT SUMMANDS

GS is a 2-functor $\mathrm{Rep}^{\Omega}\,\mathfrak{sl}_n \longrightarrow \mathrm{SingSBim}$

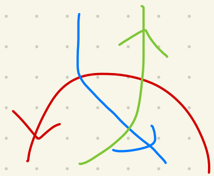
full + faithful to degree ≤ 0 maps

$\lambda \longmapsto \hat{\lambda}$ $\longleftarrow$ a maximal finitary parabolic, $W_{\hat{\lambda}} \cong W_{fin}$

${}_{\bar{1}}(\mathbb{C}^n)_{\bar{0}} \longmapsto \mathrm{Res}^{\widehat{0,1}}_{\hat{1}} \otimes \mathrm{Ind}^{\widehat{0,1}}_{\hat{0}}$

${}_{\bar{k}}(\Lambda^k \mathbb{C}^n)_{\bar{0}} \longmapsto \mathrm{Res}^{\widehat{0,k}}_{\hat{k}} \otimes \mathrm{Ind}^{\widehat{0,k}}_{\hat{0}}$

wedge product

$\Lambda^k \mathbb{C}^n \otimes \Lambda^l \mathbb{C}^n \to \Lambda^{k+l} \mathbb{C}^n \longmapsto$  $\longleftarrow$ uses Frob. structures

Elias–Snyder–Williamson

Rep thry facts become Frob ext facts.

Ex: $\tilde{A}_1$ $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. $\dim \mathbb{C}^2 = -2 \rightsquigarrow \partial_s(\alpha_t) = -2$