Conformal blocks: an overview

Chiara Damiolini

Rutgers University

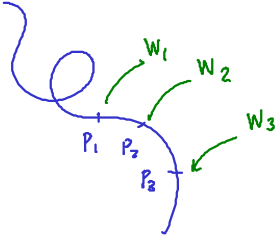
March 22 2021

Classical conformal blocks

Conformal blocks are vector spaces depending on

Geometry

C proj curve (nodal)



$p_1 \dots p_n \in C(\mathbb{C})$

$(C, \underline{p}) \in \overline{\mathcal{M}}_{g,n}(\mathbb{C})$

Representation theory

$\begin{cases} \text{simple Lie algebra } \mathfrak{g} \\ \ell \in \mathbb{Z}_{>0} \end{cases}$ $(\mathfrak{sl}_r)$

$W \in P_\ell$ = irr. repr. of $\mathfrak{g}$ of level ℓ $(\mathfrak{sl}_2$

$\forall$ every nilpotent elmt of $\mathfrak{g}$ X, $X^{\ell+1} \equiv 0$ on W

$P_\ell \updownarrow \{0, \dots, \ell\}$

$\{ W_1 \dots W_n \in P_\ell$

$\rightsquigarrow \mathbb{V}\big((C, \underline{p}), (\mathfrak{g}, \ell, \underline{W})\big)^*$ space of conformal blocks

coinvariants

The case of $C = \mathbb{P}^1$

P_i given by $t = t_i$

Fix $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$, with H_θ the highest coroot, and $X_\theta \in \mathfrak{g}^\theta$ such that $[H_\theta, X_\theta] = 2X_\theta$.

$$\mathbb{V}((\mathbb{P}^1, \underline{P}), (\mathfrak{g}, \ell, \underline{W})) = \frac{W_1 \otimes \cdots \otimes W_n}{\mathfrak{g}(W_1 \otimes \cdots \otimes W_n), \boxed{T^{\ell+1}}(W_1 \otimes \cdots \otimes W_n)}$$

<u>RMK</u> • $\mathbb{V}(\mathbb{P}^1, P_1 P_2 P_3)(\mathfrak{sl}_2, \ell, a\ b\ c)) = \mathbb{C}\ \delta_{a+b+c = 2m \leq 2\ell \ \& \ a,b,c \leq m}$

$P_\ell = \{0 - \ell\}$

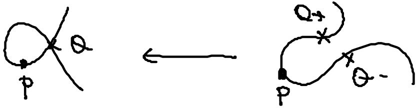
• $\mathbb{V}((\mathbb{P}^1\ P_1 P_2 P_3)(\mathfrak{g}, \ell, \text{TRIV}, W_1, W_2)) = \mathbb{C}\ \delta_{W_1 = W_2^*}$

Sheaves of conformal blocks

Theorem $\mathbb{V}((C,\underline{P}),\overbrace{(\mathfrak{g},\ell,\underline{W})}^{\text{fixed}})$ defines a vector bundle of finite rank over $\overline{\mathcal{M}}_{g,n}$ whose rank can be computed using induction on the genus. $\mathbb{V}_g(\mathfrak{g},\ell,W)$

- Vector bundles on $\overline{\mathcal{M}}_{g,n}$ $\rightsquigarrow$ can produce maps: $\overline{\mathcal{M}}_{g,n} \to \mathbb{P}^N$
  If $g=0$ $\mathbb{V}_0(\ \ )$ is globally generated!!!
- fibers identified with $H^0(\mathrm{Bun}_G, \mathcal{L}^{\otimes \ell})$ $\quad \mathrm{Lie}(G)=\mathfrak{g}$
- FACTORIZATION $\mathbb{V}(C,P,(\mathfrak{g},\ell,W)) \cong \bigoplus_{M\in P_\ell} \mathbb{V}(C_N, P\,Q_+\,Q_-, \mathfrak{g},\ell,W,M,M^*)$

$$\overline{\mathcal{M}}_{g,n} \rightsquigarrow \overline{M}_{0,n+2g} \rightsquigarrow \overline{M}_{0,3}$$



An old example

$\mathfrak{g} = \mathfrak{sl}_2 \qquad P_\ell = \{0, \ldots, \ell\} \qquad \mathbb{V}_0(a,b,c) = \mathbb{C}$ when $a+b+c = 2m \le 2\ell$, $\boxed{a,b,c \le m}$

if $\ell = 2$ $g = 0$ only non trivial v.s. $\boxed{\mathbb{V}(0,0,0) = \mathbb{V}(1\,1\,0) = \mathbb{V}(2\,2\,0) = \mathbb{V}(1\,1\,2) = \mathbb{C}}$

if $\ell = 1$ $\mathfrak{g} = \mathfrak{sl}_n$ then $\operatorname{rank} \mathbb{V}_{g}(1 \cdots 1) = n^g \, \delta_{\Sigma 1 \equiv 0 \bmod n}$

The general picture

V vertex oper. algebra of CohFT-type

$M_1 \dots M_n \in$ simple V-modules

Theorem [D-Gibney-Tarasca]

$\mathbb{V}((C,\underline{P}),(V,\underline{M}))$ defines a vector bundle of finite rank over $\overline{\mathcal{M}}_{g,n}$ $\Rightarrow$ previous theorem
whose rank can be computed using induction on the genus.

also Chern classes (using CohFT)

$\mathfrak{g}, \ell \rightsquigarrow V = L_\ell(\mathfrak{g})$ v.o.A

$W \in P_\ell \rightsquigarrow \mathcal{M}_\ell(W) \in$ V-mod

KAC

A new example: Virasoro VOA

$\mathrm{Vir}_{(p,q)}$ is a vertex operator algebra of CohFT type for $p, q \geq 2$ and $(p, q) = 1$.

When $p = 3$, $q = 4$, we have $\mathrm{Rep}(\mathrm{Vir}_{3,4}) = \{V, W_1, W_2\}$

only non zero

$$\mathbb{V}_0(W_2, W_1, W_1) = \mathbb{V}_0(V, W_1, W_1) = \mathbb{V}_0(V, W_2, W_2) = \mathbb{V}_0(V, V, V) = \mathbb{C}$$

same numerology of $\mathfrak{sl}_2$, $\ell = 2$

2 1 1 0

A new discovery

On $\overline{\mathcal{M}}_{0,4}$ we have

$$\deg(\mathbb{V}_0(\mathrm{Vir}_{3,4}; W_1^k, W_2^j)) = \begin{cases} 1 & j = k = 2 \\ 2 & j = 4 \\ -1 & k = 4 \end{cases}$$

Rank $\underline{\underline{2}}$

$$\deg(\mathbb{V}_0(L_2(\mathfrak{sl}_2); 1^k, 2^j)) = \begin{cases} 1 & j = k = 2 \\ 2 & j = 4 \\ 0 & k = 4 \end{cases}$$

thank you