Castelnuovo–Mumford regularity of matrix Schubert varieties

Oliver Pechenik
University of Waterloo

Geometry and Combinatorics from Root Systems
ICERM
22 Mar 2021
Joint with David Speyer and Anna Weigandt

The complete flag variety

The **complete flag variety** $\mathcal{F}\ell(\mathbb{C}^n)$ is the set of complete flags of nested vector subspaces

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n,$$

where $\dim V_i = i$.

Example

$$\text{Standard flag } \mathcal{SF}: \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \subset \begin{bmatrix} * \\ 0 \\ 0 \\ 0 \end{bmatrix} \subset \begin{bmatrix} * \\ * \\ 0 \\ 0 \end{bmatrix} \subset \begin{bmatrix} * \\ * \\ * \\ 0 \end{bmatrix} \subset \begin{bmatrix} * \\ * \\ * \\ * \end{bmatrix} = \mathbb{C}^4$$

Since $\mathcal{F}\ell(\mathbb{C}^n)$ has transitive action of GL_n, we can identify it with $\mathrm{GL}_n(\mathbb{C})/\mathrm{Stab}(\mathcal{SF}) = \mathrm{GL}_n(\mathbb{C})/U$, where $U =$ upper triangular matrices.

Matrix Schubert varieties

Bruhat decomposition: $\mathrm{GL}_n = \coprod_{w \in S_n} LwU$

Schubert cells: $X_w^\circ = LwU/U \subset \mathcal{F}\ell(\mathbb{C}^n)$

Schubert varieties: $X_w = \overline{X_w^\circ}$ give a complex cell decomposition of $F\ell(\mathbb{C}^n)$.

The **matrix Schubert variety** (Fulton 1992) $\tilde{X}_w = \overline{LwU} \subseteq \mathrm{Mat}(n)$ is defined by rank conditions on maximal northwest submatrices.

Castelnuovo–Mumford regularity

- R a polynomial ring, $I \subseteq R$ a homogeneous ideal
- A **free resolution** of R/I is an exact diagram of graded R-modules

$$0 \to \bigoplus_{i \in \mathbb{Z}} R(-i)^{b_i^k} \to \cdots \to \bigoplus_{i \in \mathbb{Z}} R(-i)^{b_i^0} \to R/I \to 0$$

  that is exact.
- **Minimal free resolution** simultaneously minimizes all b_i^j
- k is the **projective dimension** of R/I. For R/I Cohen–Macaulay, this is the codimension of $\operatorname{Spec} R/I$ in $\operatorname{Spec} R$.
- The **Castelnuovo–Mumford regularity** of R/I is the greatest $i - j$ such that $b_i^j \neq 0$.

Regularity and K-polynomials

- Write $(R/I)_a$ for the degree a piece of R/I. The **Hilbert series** of R/I is the formal power series

$$H(R/I;t) = \sum_{a \in \mathbb{N}} \dim_{\mathbb{C}} (R/I)_a t^a = \frac{K(R/I;t)}{(1-t)^{n^2}}.$$

- For I prime and R/I Cohen–Macaulay,

$$\mathrm{reg}(R/I) = \deg(K(R/I;t)) - \mathrm{codim}(\mathrm{Spec}\, R/I).$$

Grothendieck polynomials and K-polynomials

- Start with the **longest** permutation in S_n

$$w_0 = n\, n-1 \ldots 1 \qquad \mathfrak{G}_{w_0}(\mathbf{x}) := x_1^{n-1} x_2^{n-2} \ldots x_{n-1}$$

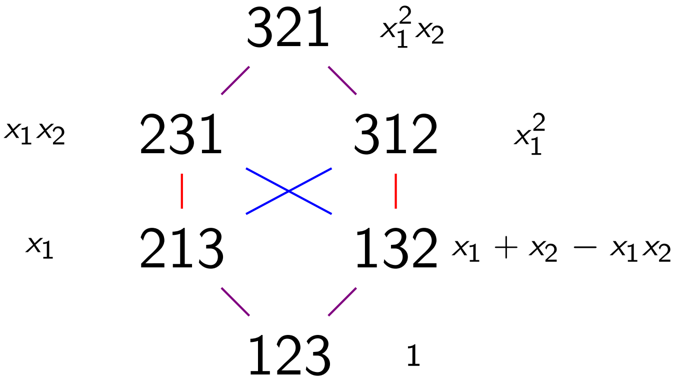
- **Grothendieck polynomials** are defined recursively by **divided difference operators:**

$$\overline{\partial_i} f := \frac{(1 - x_{i+1})f - s_i \cdot (1 - x_{i+1})f}{x_i - x_{i+1}}$$

$$\mathfrak{G}_{ws_i}(\mathbf{x}) := \overline{\partial_i}\mathfrak{G}_w(\mathbf{x}) \text{ if } w(i) > w(i+1)$$

- Setting $x_i \mapsto 1 - t$ gives the K-polynomial for the corresponding matrix Schubert variety
- The **Castelnuovo–Mumford polynomial** $\mathfrak{CM}_w(\mathbf{x})$ is the top-degree part of $\mathfrak{G}_w(\mathbf{x})$

Example Grothendieck polynomials



What is the degree of a Grothendieck polynomial?

- All of the previous was observed by Jenna Rajchgot, who then asked the title of this slide
- With Ren, Robichaux, St. Dizier, and Weigandt (2021), she gave a formula for the *Grassmannian* case

Theorem (P+Speyer+Weigandt)

For $w \in S_n$, we have $\deg \mathfrak{CM}_w(\mathbf{x}) = \mathrm{raj}(w)$, *the* **Rajchgot index** *of w.*

In particular, the Castelnuovo–Mumford regularity of the matrix Schubert variety $\tilde{X}_w$ is $\mathrm{raj}(w) - \mathrm{inv}(w)$.

Moreover, for any term order satisfying $x_1 < x_2 < \cdots < x_n$, the leading term of $\mathfrak{CM}_w(\mathbf{x})$ is a scalar multiple of the monomial $\mathbf{x}^{\mathrm{rajcode}(w)} = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$.

Rajchgot index and code

- Let $w = w(1)w(2)\cdots w(n)$
- For each k, find a longest increasing subsequence of $w(k)w(k+1)\cdots w(n)$ containing $w(k)$
- Let r_k be the number of terms from $w(k)w(k+1)\cdots w(n)$ omitted from this subsequence.
- $(r_1, \ldots, r_n) = \mathrm{rajcode}(w)$ is the **Rajchgot code** of w and its sum $\mathrm{raj}(w)$ the **Rajchgot index** of w.

Example

$w = 293417568$. A longest increasing subsequence starting from 2 is $2 \bullet 34 \bullet \bullet 568$, which omits three terms, so $r_1 = 3$. In full,

$$\mathrm{rajcode}(w) = (r_1, r_2, \ldots, r_9) = (3, 7, 2, 2, 1, 2, 0, 0, 0).$$

The leading monomial of $\mathfrak{CM}_w(\mathbf{x})$ is $x_1^3 x_2^7 x_3^2 x_4^2 x_5 x_6^2$ and the degree of $\mathfrak{CM}_w(\mathbf{x})$ is $\mathrm{raj}(w) = 17$. Since $\mathrm{inv}(w) = 12$,

$$\mathrm{reg}(\tilde{X}_w) = \mathrm{raj}(w) - \mathrm{inv}(w) = 17 - 12 = 5.$$

Unlike $\mathfrak{G}_w(\mathbf{x})$, many $\mathfrak{CM}_w(\mathbf{x})$ coincide up to scalar multiple.
Distinct $\mathfrak{CM}_w(\mathbf{x})$ are counted by Bell numbers.

Theorem (P+Speyer+Weigandt)

Double Castelnuovo–Mumford polynomials factor into **Rajchgot polynomials** *as*

$$\mathfrak{CM}_w(\mathbf{x};\mathbf{y}) = \mathfrak{R}_{\pi(w)}(\mathbf{x})\mathfrak{R}_{\pi(w^{-1})}(\mathbf{y}).$$

Moreover, for any term order satisfying

$$x_1 < x_2 < \cdots < x_n \quad \textit{and} \quad y_1 < y_2 < \cdots < y_n,$$

$\mathfrak{CM}_w(\mathbf{x};\mathbf{y})$ *has leading term exactly* $\mathbf{x}^{\mathrm{rajcode}(w)}\mathbf{y}^{\mathrm{rajcode}(w^{-1})}$

Two more characterizations of Rajchgot index

Rajchgot index can also be computed from major index on Bruhat intervals.

Theorem (P+Speyer+Weigandt)

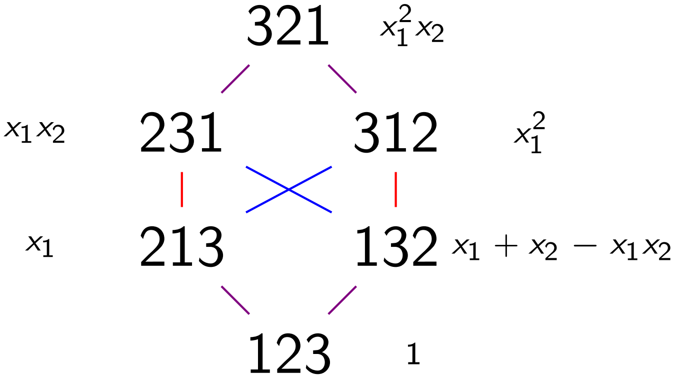
For all $w \in S_n$,

$$\mathrm{raj}(w) = \max\{\mathrm{maj}(v) : v \leq_R w\} = \max\{\mathrm{maj}(u^{-1}) : u \leq_L w\},$$

where $\leq_L$ *and* $\leq_R$ *denote the left and right weak orders, respectively.*

Idea of proof that $\deg \mathfrak{CM}_w(\mathbf{x}) = \mathrm{raj}(w)$

- Not hard to see that $\deg \mathfrak{CM}_w(\mathbf{x}) = \mathrm{raj}(w)$ for *dominant permutations* (132-avoiding)
- Also not hard for *layered permutations* (231- and 312-avoiding)
- Both $\deg \mathfrak{CM}_w(\mathbf{x})$ and $\mathrm{raj}(w)$ are weakly increasing in 2-sided weak order
- Show that every w sits between a layered permutation and a dominant permutation with the same Rajchgot index



Thank you!!