Problems about cluster varieties

There are no logical dependencies between these problems, except for that the very last problem uses the proceeding two. You certainly won't have time to work on all of them – choose something that appeals to you!

The first few problems work out properties of the space

$$\mathcal{Y} := \{(x, x', y) \in \mathbb{C}^3 \ : \ xx' = y + 1, \ y \neq 0\}.$$

Problem 1. It was claimed in the first lecture that $\mathcal{Y}$ has a deformation retract onto the subspace $|x| = |x'|$, $|y| = 1$. Verify this.

Problem 2. This problem constructs a 2-form on $\mathcal{Y}$ which will be a de Rham representative for a nonzero class in $H^2(\mathcal{Y})$.

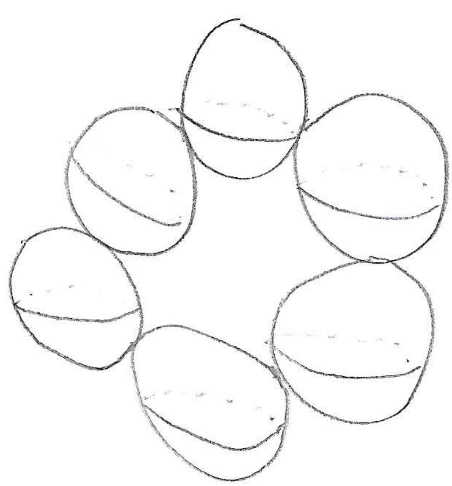
(1) Verify that $\frac{dx \wedge dy}{xy} = -\frac{dx' \wedge dy}{x'y}$.
(2) Show that this 2-form extends to a globally defined 2-form on all of $\mathcal{Y}$. This can either be done by knowing some general principles of complex analysis (Hartog's extension theorem) or by writing down an explicit formula for the extension to all of $\mathcal{Y}$.
(3) (For those who enjoy calculus.) Show that the integral of $\frac{dx \wedge dy}{xy}$ over the 2-cycle $\{|x| = |x'|, \ |y| = 1\} \subset \mathcal{Y}$ is $4\pi^2$ (or possibly $-4\pi^2$, depending how you orient this 2-cycle). Since this integral is nonzero, this verifies that $\frac{dx \wedge dy}{xy}$ is a nonzero de Rham class.

Problem 3. Define

$$\mathcal{Y}_b := \{(x, x', y) \in \mathbb{C}^3 \ : \ xx' = y^b + 1, \ y \neq 0\}.$$

Observe that this is again a cluster variety, coming from the quiver $x \xrightarrow{b} \boxed{\text{y}}$. Here the variable y is frozen, and the b indicates that there are b arrows from x to y.

Give a deformation retract from $\mathcal{Y}_b$ onto a ring of b two-dimensional spheres, as shown in the image (for $b = 6$).



Prof. Williams lectures described cluster seeds using quivers. We'll often want the alternate terminology of B-matrices. Let $(x_1, x_2, \ldots, x_n)$ be a seed with corresponding quiver Q. The B-matrix of this seed is the skew symmetric matrix where B_{ij} is #(arrows $i \to j$) − #(arrows $j \to i$). If we have n mutable variables and m frozen variables, we often use just the first n columns of this matrix, but all $n+m$ rows, making an $(n+m) \times n$ matrix called $\tilde{B}$.

The next few problems show some of the things we can do with B-matrices.

Problem 4. Let $(x_1, x_2, \ldots, x_n)$ be a seed and define the B-matrix as above. For simplicity, let's say there are no frozen variables. Set

$$\gamma = \sum_{1 \leq i < j \leq n} B_{ij} \frac{dx_i \wedge dx_j}{x_i x_j}.$$

Let $(x_1, x_2, \ldots, x_{n-1}, x'_n)$ be a neighboring cluster, with corresponding matrix B', and define γ' likewise. Show that γ and γ' define the same 2-form, on the open set where they are both well defined.

Problem 5. Let A be a cluster algebra. Suppose that we have an action (by $\mathbb{C}$-algebra automorphisms) of the group $(\mathbb{C})^*$ on A, such that, for every cluster variable x, there is some integer $d(x)$ such that $t \cdot x = t^{d(x)} x$ for $t \in \mathbb{C}^*$.

(1) Let $(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ be a cluster with n mutable and m frozen variables, and define the $\tilde{B}$ matrix as above. Let $\vec{d}$ be the row vector $(d(x_1), d(x_2), \ldots, d(x_{n+m}))$. Show that $\vec{d}\tilde{B} = 0$.

(2) (Harder) Conversely, let $\vec{d}$ be an integer vector in the left kernel of $\tilde{B}$. Show that there is an action of $\mathbb{C}^*$ on the cluster algebra as above. (Hint: give an inductive recipe for computing $d(x)$ for x not in the original seed.)

Problem 6. Let A be a cluster algebra, let y be a frozen variable and let Q be the quiver of some seed. Show that $A/(y-1)A$ is isomorphic to the cluster algebra corresponding to deleting the vertex y from Q.

Problem 7. This Problem uses Problems 5 and 6, which are independent of each other. Let A be a cluster algebra n mutable variables and m frozen variables. Let $(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ be a seed and let $\tilde{B}$ be the B-matrix. Suppose that the last row of $\tilde{B}$ is in the integer span of the other rows. Let A_1 be the cluster algebra corresponding to deleting this last row from the $\tilde{B}$-matrix (equivalently, deleting the vertex x_{n+m} from the quiver.)

Let $\mathcal{Y}$ be the cluster variety Spec A. Let $\mathcal{Y}_1 =$ Spec A_1. Show that $\mathcal{Y} \cong \mathcal{Y}_1 \times \mathbb{C}^*$.