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PM Session

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Tropical Moduli Spaces Problem Session

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The goal of these problems is to study the topology of $\Delta_{1,3}$—the moduli space of stable tropical curves of genus 1 with 3 marked points—along with some of its natural subcomplexes.

Let $\Delta_{\mathbf{G}}$ denote the locally closed cell in $\Delta_{g,n}$ parameterizing stable tropical curves whose underlying marked weighted graph is $\mathbf{G}$. Let $\Delta^{\mathrm{bm}}_{g,n} \subset \Delta_{g,n}$ denote the closure of the locus of tropical curves with *bridges* or *multiple edges*; it consists of the cells $\Delta_{\mathbf{G}}$ where $\mathbf{G}$ has a vertex with positive weight, loop, cut-vertex, bridge, pair of parallel edges, or vertex with at least two markings.

There are 7 maximal cells in $\Delta_{1,3}$, and these are precisely the cells associated to the graphs $\mathbf{G}_i$ and $\mathbf{H}_i$, for $i \in \{1, 2, 3\}$, and $\mathbf{K}$, as shown in the following figure.

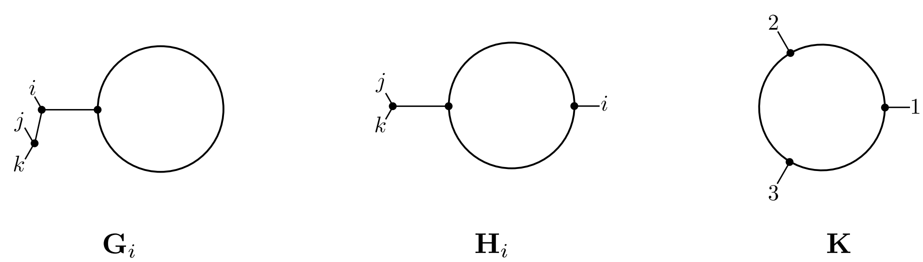


Figure 1: The markings i, j, k are distinct elements of $\{1, 2, 3\}$.

1. Let $\Sigma_i = \overline{\Delta_{\mathbf{G}_i}} \cup \overline{\Delta_{\mathbf{H}_i}}$.
   (a) Prove that $\Delta^{\mathrm{bm}}_{1,3} = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$.
   (b) Prove that each Σ_i is contractible.
   (c) Prove that $\Sigma_i \cap \Sigma_j$ is a point, for distinct $i, j \in \{1, 2, 3\}$, as is $\Sigma_1 \cap \Sigma_2 \cap \Sigma_3$.
   (d) Conclude that $\Delta^{\mathrm{bm}}_{1,3}$ is contractible.
2. Let $\mathbf{K}$ be the graph on the right in Figure 1.
   (a) Prove that $\Delta_{\mathbf{K}}$ is the only cell of $\Delta_{1,3}$ that is not contained in $\Delta^{\mathrm{bm}}_{1,3}$.
   (b) Prove that $\Delta_{1,3}/\Delta^{\mathrm{bm}}_{1,3}$ is homeomorphic to S^2.
   (c) Conclude that $\Delta_{1,3}$ is homotopy equivalent to S^2.

Extra Credit

The locus $\Delta_{g,n}^{\mathrm{bm}}$ is only one of several natural subcomplexes of $\Delta_{g,n}$ that are similarly useful.

- Let $\Delta_{g,n}^{\mathrm{lw}} \subset \Delta_{g,n}$ be the subcomplex with cells $\Delta_{\mathbf{G}}$ such that $\mathbf{G}$ has a *loop* or a vertex with positive *weight*.
- Let $\Delta_{g,n}^{\mathrm{rep}} \subset \Delta_{g,n}$ be the subcomplex with cells $\Delta_{\mathbf{G}}$ such that $\mathbf{G}$ has a *repeated marking*, i.e., a vertex with at least two markings.
- Let $\Delta_{g,n}^{\mathrm{br}} \subset \Delta_{g,n}$ be the subcomplex obtained as the closure of the locus of tropical curves with *bridges*. Its cells are those $\Delta_{\mathbf{G}}$ where $\mathbf{G}$ has a vertex with positive weight, loop, cut-vertex, bridge, or vertex with at least two markings.

3. Prove that $\Delta_{1,3}^{\mathrm{rep}}$ and $\Delta_{1,3}^{\mathrm{br}}$ are equal to $\Delta_{1,3}^{\mathrm{bm}}$.

4. Consider $\Delta_{1,3}^{\mathrm{lw}}$.

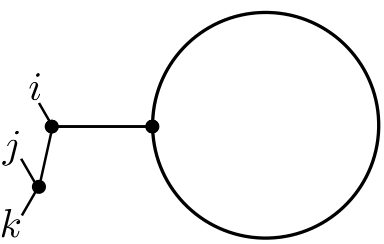
   (a) Prove that $\Delta_{1,3}^{\mathrm{lw}} = \overline{\Delta_{\mathbf{G}_1}} \cup \overline{\Delta_{\mathbf{G}_2}} \cup \overline{\Delta_{\mathbf{G}_3}}$.

   (b) Prove that each $\overline{\Delta_{\mathbf{G}_i}}$ is contractible.

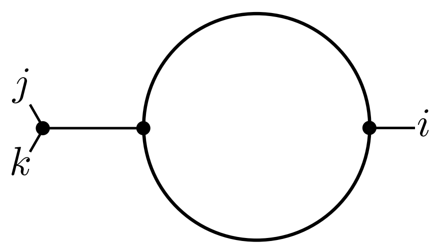
   (c) Prove that $\overline{\Delta_{\mathbf{G}_i}} \cap \overline{\Delta_{\mathbf{G}_j}}$ is a point, for distinct $i, j \in \{1, 2, 3\}$, as is $\overline{\Delta_{\mathbf{G}_1}} \cap \overline{\Delta_{\mathbf{G}_2}} \cap \overline{\Delta_{\mathbf{G}_3}}$.

   (d) Conclude that $\Delta_{1,3}^{\mathrm{lw}}$ is contractible.

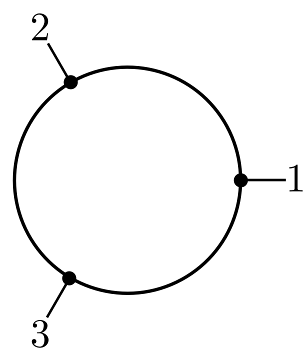
5. Draw $\Delta_{1,3}$.



$\mathbf{G}_i$



$\mathbf{H}_i$



$\mathbf{K}$