Thursday, February 4, 2021

AM Session

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ICERM INTRODUCTION TO TROPICAL CURVES EXERCISES

MADELINE BRANDT
WITH T.A.S SHIYUE LI, YELENA MANDELSHTAM, CLAUDIA YUN

(1) Let $a,\ b,\ c \in \mathbb{R}$.
   (a) Draw the graph of the tropical polynomial $f(x) = \min(a + x, b)$. What is the tropical hypersurface of $\min(a + x, b)$?
   (b) Draw the graph of the tropical polynomial $g(x) = \min(a + 2x, b + x, c)$. There are two possibilities– write a condition in terms of a, b, and c that distinguishes the two possibilities. What is the tropical hypersurface of $\min(a + 2x, b + x, c)$ in each case?

(2) Let $K = \mathbb{C}\{\{t\}\}$ with the valuation $\mathrm{val}_K(c_0 t^{a_0} + c_1 t^{a_1} + \cdots) = a_0$ and let
$$f(x, y) = t^2 + tx + tx^2 + t^3x^3 + ty + xy + tx^2y + ty^2 + txy^2 + t^3y^3.$$
   (a) Compute $\mathrm{trop}(f)$.
   (b) Use any method (computers permitted but not necessary) to compute trop(V(f)).
   (c) The curve $V(f)$ is an elliptic curve. Every elliptic curve over $\mathbb{C}\{\{t\}\}$ can be re-embedded so that its equation is of the form $y^2 = x^3 + ax + b$ for some $a, b \in \mathbb{C}\{\{t\}\}$ (this is called *Weierstrasss form*). Sketch the possible tropicalizations of elliptic curves in Weierstrass form.

(3) How many combinatorial types of quadratic curves are there? Meaning, sketch the possible tropical hypersurfaces of the polynomial in variables x and y given by
$$ax^2 + bx + c + dy + exy + fy^2$$
for $a, b, c, d, e, f \in \mathbb{C}\{\{t\}\}^*$.

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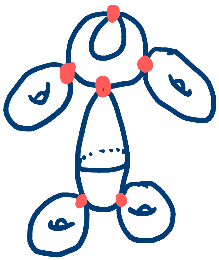
Introduction to Tropical Geometry at ICERM

February 4 2021

Madeline Brandt

Overview: 2 friends

Tropical geometry tells us how to relate these friends:

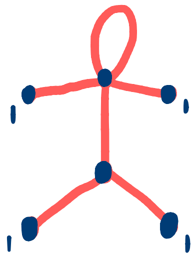


Algebraic Geometry

Algebraic Varieties

$\{x \in K^n \mid f_1(x) = \cdots = f_\ell(x) = 0\}$

$\xrightarrow{\text{trop}}$



Combinatorics

tropical varieties/ polyhedral complexes

Today:

Part I: Embedded tropical geometry via curves in the plane

Part II: Abstract tropical geometry & the two friends.

Geometry over Non-Archimedean fields

Tropical Geometry deals with varieties over Non-Archimedean fields. These fields have a norm that behaves very differently from the Archimedean norm on $\mathbb{C}$.

Definition. $(K, |\cdot|)$ is an Archimedean field if it satisfies the Archimedean Axiom:

for any $x \in K^*$, there is an $n \in \mathbb{N}$ such that $|nx| > 1$.

This axiom feels natural and familiar — but $\mathbb{R}$ and $\mathbb{C}$ are the only complete Archimedean fields (Ostrowski's theorem)

A non-Archimedean field K is one with a norm which fails this axiom.

It comes with a function called the valuation

$$\mathrm{val}_K : K \to \mathbb{R} \cup \{\infty\}$$

$$a \mapsto -\log(|a|) \quad a \neq 0, \qquad 0 \mapsto \infty$$

Example. The trivial valuation on any K is: $0 \mapsto \infty$, $K^* \mapsto 0$

Example. The Puiseux series $\mathbb{C}\{\{t\}\}$ is:

$$\left\{ c(t) = c_1 t^{a_1} + c_2 t^{a_2} + \cdots \;\middle|\; \begin{array}{l} c_1 \neq 0, \quad c_i \in \mathbb{C} \\ a_i \text{ an } \nearrow \text{ seq. in } \mathbb{Q} \\ \text{w/ common denominator} \end{array} \right\} \cup \{0\}$$

norm: $|c(t)| = (1/e)^{a_1}$

val: $\mathrm{val}_K(c(t)) = -\log(|c(t)|) = -\log((1/e)^{a_1})$

$$= -a_1 \log(1/e) = a_1$$

* alg. closed *

Embedded tropicalization

How to find the embedded tropicalization of a hypersurface over a non-Archimedean field.

Definition. Given a Laurent polynomial

$$f(x) = \sum_{a \in \mathbb{Z}^n} c_a x^a \quad \in \quad K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

its tropicalization $\mathrm{trop}(f) : \mathbb{R}^n \to \mathbb{R}$ is

$$\mathrm{trop}(f)(x) = \min_{a \in \mathbb{Z}^n} \left(\mathrm{val}_K(c_a) + a \cdot x \right)$$

Just as we can associate a variety to f, which would be a hypersurface in $(K^*)^n$, we will associate a tropical variety to $\mathrm{trop}(f)$.

Definition. The tropical hypersurface $\mathrm{trop}(V(f))$ is the set

$$\left\{ w \in \mathbb{R}^n \;\middle|\; \text{the minimum in } \mathrm{trop}(f)(w) \text{ is achieved at least twice} \right\}$$

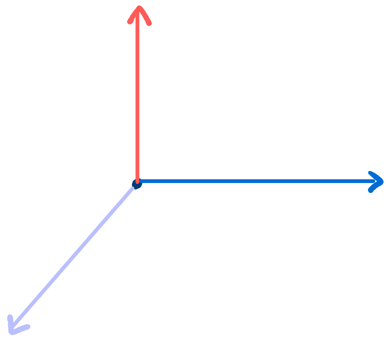
Example [tropical line]. Let $f = x + y + 1 \in \mathbb{C}\{\{t\}\}[x,y]$.

Then $\mathrm{trop}(f) : \mathbb{R}^2 \to \mathbb{R}$, and

$$\begin{aligned} \mathrm{trop}(f)(w_1, w_2) &= \min\big(\mathrm{val}_K(1) + (1,0)\cdot(w_1,w_2), \\ &\qquad\quad \mathrm{val}_K(1) + (0,1)\cdot(w_1,w_2), \\ &\qquad\quad \mathrm{val}_K(1)\big) \\ &= \min(w_1, w_2, 0) \end{aligned}$$

make 3 cases:

- w_1 & 0 are min: $w_1 = 0,\ w_2 \geq 0$
- w_2 & 0 are min: $w_2 = 0,\ w_1 \geq 0$
- w_1 & w_2 are min: $w_1 = w_2,\ \ w_1 \leq 0$.



tropical line

(note: this would be very inefficient if you had many terms)

Theorem [Kapranov ⇝ Fundamental Thm]

The set trop(V(f)) is the same as

$$\overline{\{(\mathrm{val}_K(y_1), \dots, \mathrm{val}_K(y_n)) \mid (y_1, \dots, y_n) \in V(f)\}}$$

Example [tropical line]. Let $f = x + y + 1 \in \mathbb{C}\{\{t\}\}[x,y]$.

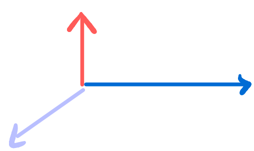
$y = -x-1 \implies$ all solns are $(x, -x-1)$.

Kapranov Theorem $\implies \mathrm{trop}(V(f)) = \{(\mathrm{val}_K(x), \mathrm{val}_K(-x-1)) \mid x \in K\}$

$\mathrm{val}_K(x) > 0$: $\mathrm{val}_K(-x-1) = 0$

$\mathrm{val}_K(x) = 0$: $\mathrm{val}_K(-x-1) \geq 0$

$\mathrm{val}_K(x) < 0$: $\mathrm{val}_K(-x-1) = \mathrm{val}_K(x)$



Remark. What kind of object is a tropical variety?

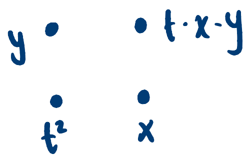
A lot can be said about its structure [Structure Theorem]

Proposition [Practical method for curves in the plane by hand]

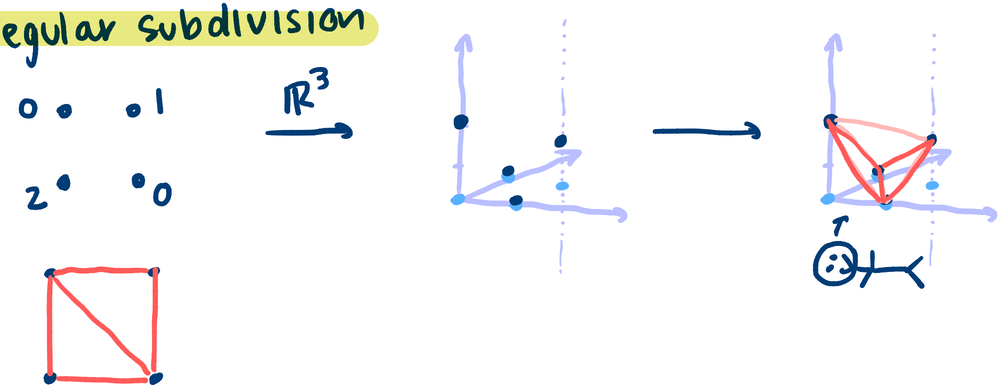
Let $f \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. The tropical hypersurface $\operatorname{trop}(V(f))$ is the $(n-1)$-skeleton of the polyhedral complex dual to the regular subdivision of the Newton polytope of f induced by the weights $\operatorname{val}(c_u)$

Example. Let $f(x,y) = t \cdot x \cdot y + x + y + t^2 \in \mathbb{C}\{\{t\}\}[x,y]$

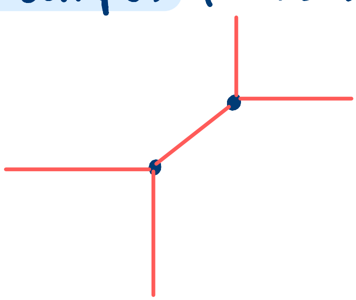
1. Newton Polytope: one lattice point per monomial



2. Regular subdivision



3. dual complex (rotate by 180°)



Exercises.

Problem 1. Let $k = \mathbb{C}\{\{t\}\}$, and let

$$f(x,y) = t^3 + tx + tx^2 + t^3x^3 + ty + xy + tx^2y + ty^2 + txy^2 + t^3y^3$$

1. Compute $\operatorname{trop}(f)$.
2. Using any method you like compute $\operatorname{trop}(V(f))$.
3. $V(f)$ is an elliptic curve. Every elliptic curve over $\mathbb{C}\{\{t\}\}$ can be re-embedded so that its equation is of the form $y^2 = x^3 + ax + b$ for $a, b \in \mathbb{C}\{\{t\}\}$ (Weierstrass form). What are all the possibilities for tropicalizations of elliptic curves in Weierstrass form?

Problem 2. Let $a \in \mathbb{C}\{\{t\}\}^*$ and $b, c \in \mathbb{C}\{\{t\}\}$.

1. Determine $\operatorname{trop}(V(a \cdot x + b))$.
2. Determine $\operatorname{trop}(V(a \cdot x^2 + bx + c))$.

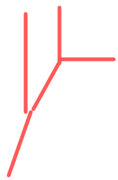
Problem 3. How many combinatorial types of tropical quadratic curves are there? ie, tropicalizations of

$$0 = ax^2 + bx + c + dy + exy + fy^2$$

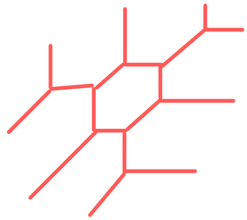
for $a, \ldots, f \in \mathbb{C}\{\{t\}\}^*$.

Abstract Tropicalization

From exercise 1 in the problem session, you saw that a curve can have different embedded tropicalizations depending on the embedding:



vs

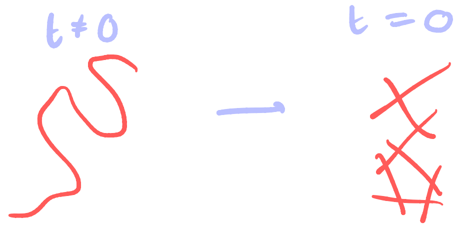


Theorem [Chan-Sturmfels] Every elliptic curve with $val_K(j) < 0$ has an embedding such that the embedded tropicalization is a honeycomb.

Question. How do we associate an "intrinsic" tropical object to a curve?

Abstract Tropicalization

A curve over $\mathbb{C}\{\{t\}\}$ can be thought of as a family of curves depending on a parameter t.



Informally we can think of this parameter as "going to zero", and when $t=0$, we observe some possibly singular behavior.

Notation.

Let $R = \{ f \in K \mid \mathrm{val}_K(f) \geq 0 \}$.

This is a local ring with unique maximal ideal $\mathfrak{m} = \{ f \in K \mid \mathrm{val}_K(f) > 0 \}$.

Spec(R) is a topological space with 2 points:

$$\{ \underset{\substack{\uparrow \\ \mathfrak{m} \\ \text{closed}}}{\bullet} \,,\; \underset{\substack{\uparrow \\ 0 \\ \text{open}}}{\bullet} \}$$

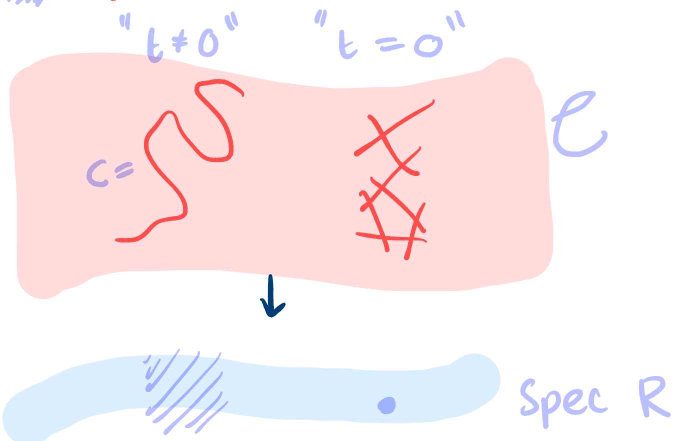
Models of Curves

More formally, we need models of curves.

Let K be complete w.r.t Val_K (e.g. the completion of $\mathbb{Q}\{t\}$)

Suppose C is a smooth and proper curve over K.

Definition. A model $\mathcal{C}$ of C over K is a proper and flat scheme over R whose fiber over (generic fiber) is isomorphic to C.



$\mathcal{C}$ is called semistable if the fiber over (special fiber) is reduced, has at worst nodal singularities, and every rational component has at least 2 singular points.

combinatorial

Remark. By the semistable reduction theorem we are always guaranteed that a semistable model for C exists.

The semistable reduction theorem guarantees that we can put curves into a combinatorially tractible form.

You can think of this as a "good" embedding from the perspective of tropical geometry.

Example. Consider $y^2 = x^3 + x^2 + t^4$ over $\mathbb{C}\{\{t\}\}$.

Dual Graphs

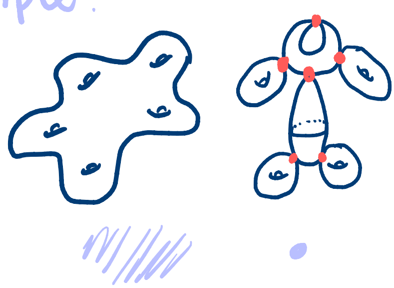
Let C be a curve over K, and let $\mathcal{C}$ be a semistable model.

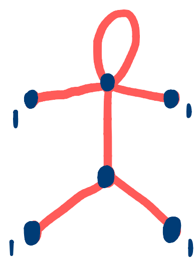
The abstract tropicalization Γ of $\mathcal{C}$ is a metric graph with:

- vertices $\leftrightsquigarrow$ irreducible components of the special fiber
- edges $\leftrightsquigarrow$ nodes of the special fiber
- vertex weights $\leftrightsquigarrow$ genus of the component
- edge lengths $\leftrightsquigarrow$ deformation parameter at each node

(locally: $xy - f$ for $f \in R$. $\mathrm{val}_K(f)$ is length)

Example.





Example. $y^2 = x^3 + x^2 + t^4$

minimal skeletons

A tuple $(G=(V,E), w, \ell)$ is called a **tropical curve**.

$w: V \to \mathbb{N}$ $\quad$ $\ell: E \to \mathbb{R}_{\geq 0}$

The **genus** of a tropical curve is $\sum_{v \in V} w(v) + |E| - |V| + 1$

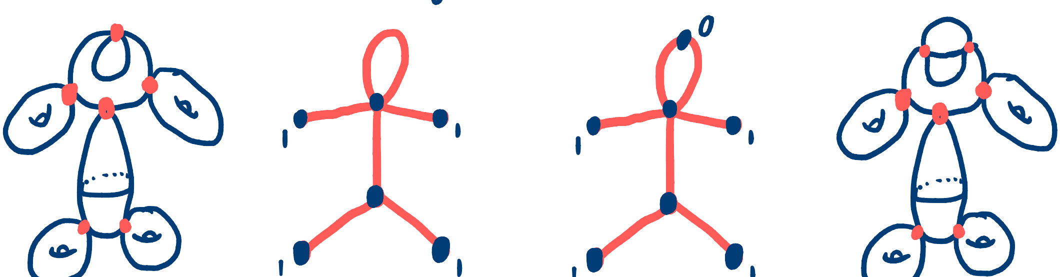
We say two tropical curves (G, w, ℓ) are **isomorphic** if one can be obtained from the other by

- graph automorphisms 
- contracting weight 0 leaves 

- "erasing" valence 2 weight 0 vertices 
- contracting length 0 edges. 

Remark. Different semistable models for a curve C will have isomorphic tropical curves. (ie, "tropical curve of C" is well-defined).

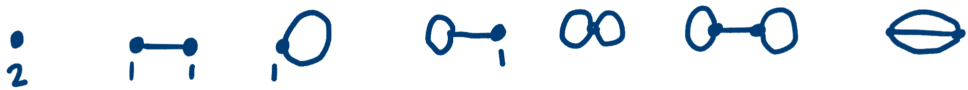
Example. The following tropical curves are isomorphic



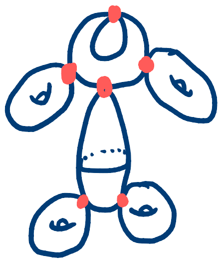
Proposition. Every tropical curve of genus ≥ 2 has a **minimal skeleton**, which is a $(G=(V,E), \ell, w)$ with

- no vertices of weight 0 and degree ≤ 2
- no edges of length 0

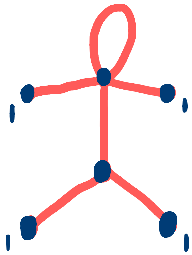
Example. Here are all combinatorial types (i.e., forget ℓ) of tropical curves of genus 2:



2 friends

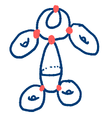
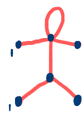


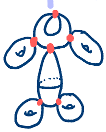
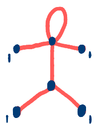
Algebraic Geometry



Combinatorics

How are the friends related?

- the abstract tropicalization of  is  .

- "good" or faithful embedded tropicalizations of  contain  .

Remark. Today I focused on curves... but this can be done for higher dimensional varieties!!


Thank You!