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AM Session

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QUANTUM COHOMOLOGY OF HOMOGENEOUS SPACES EXERCISE SESSION

NICOLAS PERRIN

1. Warm-up

Exercise 1.1. *Let* $X = \mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$.

(1) *Prove that* $\deg(q) = n+1$ *(you can use the canonical divisor but also the dimension of the space of lines!).*
(2) *Let* $B \in \mathrm{GL}_{n+1}(\mathbb{C})$ *be the subgroup of upper triangular matrices. Compute the* B*-orbits in* X *and prove that their closures are the* B*-stable linear subspaces in* X*. These closures are the Schubert varieties* $(\sigma^{(i)})_{i\in[0,n]}$.
(3) *Describe Poincaré duality for these Schubert classes.*
(4) *Let* $H \subset X$ *be the* B*-stable hyperplane and* $\mathrm{pt} \in X$ *be the* B*-stable point and let* $h = [H] = \sigma^{(1)}$ *and* $[\mathrm{pt}] = \sigma^{(n)}$*. Compute* $\langle h, [\mathrm{pt}], [\mathrm{pt}]\rangle_{X,1}$.
(5) *Compute all products* $h \star \sigma^{(i)}$ *for* $i \in [0, n]$.
(6) *Compute all quantum products* $\sigma^{i)} \star \sigma^{(j)}$ *of Schubert varieties for* $i, j \in [0, n]$.
(7) *Prove the equality* $\mathrm{QH}(X) = \mathbb{C}[h, q]/(h^{n+1} - q)$.

ADVANCED LEVEL

In this exercise session, I propose three quantum cohomology exercise packages: Grassmannian, Geometry and Presentation and semisimplicity that focus on different aspects. Note that grassmannians are so nice that they show up everywhere.

(1) The first one, Grassmannian, focuses on grassmannians and combinatorics of partitions.
(2) The second one, Geometry, deals with some generalisations of the easy projective geometric fact that there is a unique line passing through two given points.
(3) The third one, Presentation and semisimplicity, concentrates on giving presentations of the quantum cohomology ring and deals with two examples: grassmannnians and a non semisimple case.

I expect that you **choose one package and concentrate on it**. Later on you can discuss the problems in the other packages with the groups that had chosen those exercises.

At the very end, I also give another easy exercise that is good to discuss during (unfortunately virtual) coffee breaks.

Have fun!

2. Grassmannian

Reminder: A partition is a non increasing sequence $\lambda = (\lambda_i)_{i\in\mathbb{Z}_{\geq 0}}$ of non-negative integers. The partition is in the $k \times (n-k)$ rectangle if $\lambda_i = 0$ for $i > k$ and $\lambda_1 \leq n-k$. We then write $\lambda \subset \mathcal{P}_{k,n}$. For $p \in \mathbb{Z}_{\geq 0}$, we write (p) for the partition λ with $\lambda_1 = p$ and $\lambda_2 = 0$ and set

$$M_\lambda^p = \left\{ \begin{array}{c} \text{partitions obtained from } \lambda \text{ by adding } p \text{ boxes,} \\ \text{with no two boxes in the same column} \end{array} \right\}.$$

If $X = \mathrm{Gr}(k,n)$, then Schubert varieties are indexed by partitions in the $k \times (n-k)$ rectangle: $(\sigma^\lambda)_{\lambda\subset\mathcal{P}_{k,n}}$. Recall quantum Pieri formula for the grassmannian

$$\sigma^{(p)} \star \sigma^\lambda = \sum_\mu \sigma^\mu + q \sum_\nu \sigma^{\bar{\nu}},$$

where μ runs over all partitions with $\mu \subset M_\lambda^p$ and $\mu \subset \mathcal{P}_{k,n}$, where ν runs over all partitions $\nu \in M_\lambda^p$ with $\nu \subset \mathcal{P}_{k+1,n+1}$ and $\bar{\nu}$ is obtained from ν by removing a full row of length $n-k$ and a full column of length $k+1$ (so you remove n boxes).

For $X = \mathrm{Gr}(k,n)$, recall that Giambelli formula in $H^*(X,\mathbb{C})$ is given as follows: $\sigma^\lambda = \det(\sigma^{(\lambda_i+j-i)})_{i,j\in[1,k]}$. In Exercise 2.1, we assume that Giambelli formula is known in cohomology and prove it for quantum cohmology so in $\mathrm{QH}(X)$.

ex-grass

Exercise 2.1. (1) *Check using Pieri formula, that in the Chevalley formula $\sigma^{(1)} \star \sigma^\lambda$, there is at most one quantum term. Give a condition on λ for the existence of that term and describe this quantum part in terms of λ.*

(2) *Assume that $X = \mathrm{Gr}(2,4)$. Check the following formulas (recall that $\sigma^{(0)} = 1$, $\sigma^{(p)} = 0$ for $p < 0$ and $\sigma^{(p)} = 0$ for $p > 2$).).*

$$\sigma^{(1,1)} = \det\begin{pmatrix} \sigma^{(1)} & \sigma^{(2)} \\ \sigma^{(0)} & \sigma^{(1)} \end{pmatrix}, \sigma^{(2,1)} = \det\begin{pmatrix} \sigma^{(2)} & \sigma^{(3)} \\ \sigma^{(0)} & \sigma^{(1)} \end{pmatrix} \text{ and } \sigma^{(2,2)} = \det\begin{pmatrix} \sigma^{(2)} & \sigma^{(3)} \\ \sigma^{(1)} & \sigma^{(2)} \end{pmatrix}.$$

(3) *More generally, deduce from Pieri formula that Giambelli formula stays unchanged for quantum cohomology: $\sigma^\lambda = \det(\sigma^{(\lambda_i+j-i)})_{i,j\in[1,k]}$.*

3. Geometry

Reminder: a degree d map $f : \mathbb{P}^1 \to \mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$ is given by

$$f([u:v]) = [f_0(u,v), \cdots, f_n(u,v)],$$

where $f_0, \cdots, f_n$ are homogeneous polynomials of degree d with no common factor. For example, maps of degree 1 have lines for images while for degree 2 it maps $\mathbb{P}^1$ to a quadric. In particular it is contained in a plane. More generally, the image of a degree d map will be contained in a linear subspace of dimension d in $\mathbb{P}^n$. It is easy to find this space $\mathbb{P}^d$ as follows: write $f([u:v]) = [u^d a_d + u^{d-1} v a_{d-1} + \cdots + v^d a_0]$ with $a_i \in \mathbb{C}^{n+1}$. Then we have $f(\mathbb{P}^1) \subset \mathbb{P}(\langle a_d, \cdots, a_0\rangle)$.

ex-geo

Exercise 3.1. *In this exercise Q_n denotes a smooth n-dimensional quadric (see also Exercise 5.1) and for X homogeneous, we choose $x_1, x_2, x_3 \in X$ in general position.*

(1) *Let $X = \mathbb{P}^1$. Check that there is a unique degree 1 map $f : \mathbb{P}^1 \to \mathbb{P}^1$ with $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.*

(2) *Let* $X = Q_n$. *Check that there is a unique degree* 2 *map* $f : \mathbb{P}^1 \to X$ *with* $f(0) = x_1$, $f(1) = x_2$ *and* $f(\infty) = x_3$.
(3) *Let* $X = \mathrm{Gr}(d, 2d) \subset \mathbb{P}(\Lambda^d \mathbb{C}^{2d})$ *in its Plücker embedding. Check that there is a unique degree* d *map* $f : \mathbb{P}^1 \to X$ *with* $f(0) = x_1$, $f(1) = x_2$ *and* $f(\infty) = x_3$. *Hint: consider the pull-back* $f^*\mathcal{U}$ *of the universal subbundle and prove that* $f^*\mathcal{U} = \mathcal{O}_{\mathbb{P}^1}(-1)^d$.

Note that for $d = 2$, *the grassmannian* $\mathrm{Gr}(d, 2d) = \mathrm{Gr}(2, 4)$ *is a quadric.*

4. Presentation and semisimplicity

Reminder: a finite dimensional complex algebra A is semisimple if it is a product of fields or equivalently if A is reduced. It is also equivalent to ask that $\mathrm{Spec}(A)$ is smooth (a finite union of reduced points). If A has a presentation, another technique to prove (non)semisimplicity is to use the Jacobian criterium to prove (non)smoothness.

Exercise 4.1. *Let* $x_1, \cdots, x_n$ *be variables and recall that we have the equality* $\mathbb{C}[x_1, \cdots, x_n]_+^{\mathfrak{S}_n} = (e_1, \cdots, e_n)$ *where* e_i *is the* i*-th symmetric function in* $x_1, \cdots, x_n$. *For* $\lambda \in \mathbb{C} \setminus \{0\}$, *set*

$$I_\lambda = (e_1, \cdots, e_{n-1}, e_n - \lambda).$$

(1) *Prove that* $Z_\lambda = \mathrm{Spec}\left(\mathbb{C}[x_1, \cdots, x_n]/I_\lambda\right)$ *is a free* $\mathfrak{S}_n$*-orbit and is reduced.*
(2) *Deduce that* $Z_\lambda^k = \mathrm{Spec}\left(\mathbb{C}[x_1, \cdots, x_n]^{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}/I_\lambda\right)$ *is reduced.*
(3) *Deduce that* $\mathrm{QH}(\mathrm{Gr}(k, n))_{q=1}$ *is semisimple.*

Exercise 4.2. *Let* $X = \mathrm{IG}(2, 6)$ *the grassmannian of isotropic* 2*-dimensional subspaces in* $\mathbb{C}^6$ *endowed with a symplectic form. Representation theory tells you that there is a presentation*

$$H^*(X, \mathbb{C}) = \mathbb{C}[x_1, x_2, x_3]^G/(E_1, E_2, E_3)$$

with $\deg(x_i) = 1$ *for* $i \in [1, 3]$, *where* G *is the group generated by the transposition* $x_1 \leftrightarrow x_2$ *and the involution* $x_3 \mapsto -x_3$ *and* E_i *is the* i*-th symmetric function in* x_1^2, x_2^2, x_3^2. *Recall finaly that* $\mathrm{Pic}(X) = \mathbb{Z}$ *and* $\deg(q) = 5$.

(1) *Give the possible deformations* $\widetilde{E}_1$, $\widetilde{E}_2$ *and* $\widetilde{E}_3$ *for the presentation of* $\mathrm{QH}(X)$.
(2) *Prove that* $\mathrm{QH}(X)_{q=1}$ *is not semisimple (or equivalently that* $\mathrm{Spec}(\mathrm{QH}(X)_{q=1})$ *is singular).*

5. Coffee break gym

This last exercise is rather easy and can be done in small breaks when you want to practise your quantum cohomology daily gym!

ex-quad **Exercise 5.1.** *Let* $X = Q_n \subset \mathbb{P}(\mathbb{C}^{n+2})$ *be a smooth quadric of dimension* n *defined by the equation*

$$\sum_{2i \leq n+3} x_i x_{n+3-i} = 0.$$

(1) *Prove that* $\deg(q) = n$ *(you can use the canonical divisor but also the dimension of the space of lines or conics, cf. Exercise 3.1.(2)).*

(2) *Let* $B \in \mathrm{SO}_{n+2}(\mathbb{C})$ *be the subgroup of upper triangular matrices. Compute the* B*-orbits in* X *and prove that their closures, the Schubert varieties, are given as follows: for each* $k \in [0, n]$ *with* $2k \neq n$*, there is a unique Schubert variety of codimension* k*, denoted by* $X^{(k)}$ *and we have*
- $X^{(k)} = X \cap V(x_{n+2}, \cdots, x_{n+2-(k-1)})$ *for* $2k < n$
- $X^{(k)} = X \cap V(x_{n+2}, \cdots, x_{n-(k-1)})$ *for* $2k > n$

For $n = 2p$ *even and* $k = p$*, prove that there are two Schubert varieties* $X^{(p)}$ *and* $Y^{(p)}$ *of codimension* p *given by*

$$X^{(p)} = X \cap V(x_{2p+2}, \cdots, x_{p+2}) \text{ and } Y^{(p)} = X \cap V(x_{2p+2}, \cdots, x_{p+3}, x_{p+1}).$$

(3) *Denote the Schubert classes as follows:* $\sigma^{k)} = [X^{(k)}]$ *for* $k \in [0, n]$ *and* $\tau^{(p)} = [Y^{(p)}]$ *for* $n2p$*. These closures are the Schubert varieties* $(\sigma^{(i)})_{i \in [0,n]}$*. Describe Poincaré duality for these Schubert classes (for* n *even, this will depend on the class of* n *in* $\mathbb{Z}/4\mathbb{Z}$*).*

(4) *Let* $h = \sigma^{(1)}$ *and* $[\mathrm{pt}] = \sigma^{(n)}$*. Compute* $\langle [\mathrm{pt}], [\mathrm{pt}], [\mathrm{pt}] \rangle_{X,1}$.

(5) *Compute all products* $h \star \sigma^{(i)}$ *for* $i \in [0, n]$ *and* $h \star \tau^{(p)}$ *for* $n = 2p$*.*

(6) *Compute all quantum products of Schubert varieties.*

(7) *Prove the equality* $\mathrm{QH}(X) = \mathbb{C}[h, q]/(h^n - qh)$*.*

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