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PM Session

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EXERCISES FOR "INTRODUCTION TO MATROIDS" (ICERM)

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EXERCISES

Exercise 1. Matroid basics

Let M be a matroid on $E = \{0, 1, 2, 3\}$ with bases $\mathscr{B} = \binom{E}{3}$ (that is, every 3-subset of E is a basis). Or, if you'd like something slightly more involved, let M be a matroid on $E = \{0, 1, 2, 3, 4\}$ whose bases are $\binom{E}{3} \setminus \{\{0, 1, 2\}, \{2, 3, 4\}\}$.

(a) This matroid is graphical; which graph is it? Compute the chromatic polynomial of the graph.

(b) This matroid is linear; write down a set of four concrete vectors in $L^\vee = \mathbb{C}^3$ that realize this matroid. Draw a pictorial model of the associated projective hyperplane arrangement.

(c) Write down the lattice of flats of M, and compute the characteristic polynomial $\chi_M(q)$, and check that it agrees with (a).

Exercise 2. Wonderful compactifications

Let M be a matroid on $E = \{0, 1, 2, 3\}$ with bases $\mathscr{B} = \binom{E}{3}$ (that is, every 3-subset of E is a basis). In the previous exercise, you realized this matroid as a linear matroid associated to a linear subspace $L \subset \mathbb{C}^4$.

(a) Draw (or envision) a pictorial model of the boundary of the wonderful compactification W_L.

(b) Verify that $(\deg(\alpha^2), \deg(\alpha\beta), \deg(\beta^2))$ gives the unsigned coefficients of $\chi_M(q)/(q-1)$.

(c) Compute the defining ideal of the closure of the image of $\mathbb{P}L \subset \mathbb{P}^3$ under the Cremona transformation $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$. It is isomorphic to the Cayley nodal cubic surface (google it for a picture!). How does its degree agree with the computation in (b)? This surface has six lines and four singular points; can you see where they come from?

Exercise 3. Hodge-Riemann relations for rank 3 matroids

Let M be a loopless matroid of rank 3 on a ground set $E = \{0, 1, \ldots, n\}$. For each flat F of M with $\mathrm{rk}_M(F) = 2$, define an element $h_F \in A^1(M)$ by $h_F = \alpha - x_F$.

(a) Let $\{F_1, \ldots, F_m\}$ be the set of rank 2 flats of M. Show that $\{\alpha, h_{F_1}, h_{F_2}, \ldots, h_{F_m}\}$ forms a basis of $A^1(M)$.

(b) Suppose M is a linear matroid realized by $\mathbb{P}L \subset \mathbb{P}^n$. Convince yourself that $h_F \in A^1(M)$, considered as a divisor class on W_L, is represented by the strict transform of a general line in $\mathbb{P}L$ containing the point $\mathbb{P}L_F$.

(c) Suppose M is a linear matroid. Using (b), compute the matrix of the symmetric bilinear pairing $A^1(M) \times A^1(M) \to \mathbb{R}$ given by $(x, y) \mapsto \deg(x \cdot y)$, with respect to the basis of $A^1(M)$ in (a). (You may remove the linear matroid condition if you purely work algebraically with $A^\bullet(M)$). What is the signature of this symmetric matrix, and how does it agree with the Hodge index theorem for surfaces?