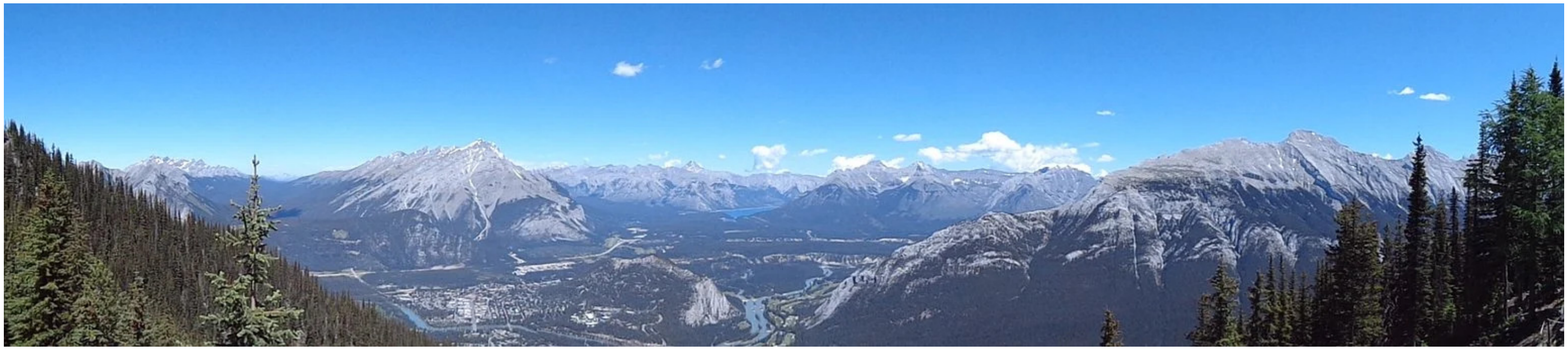
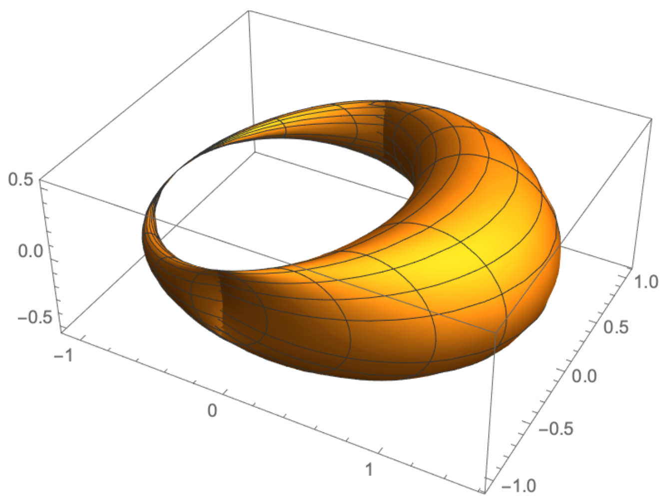
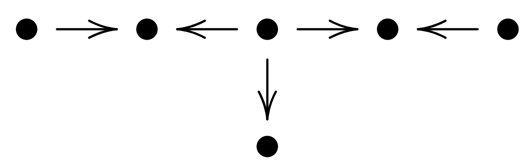


Geometry of cluster varieties II

David E Speyer





H^0	H^1	H^2	H^3	H^4	H^5	H^6
1		q^2		q^4		q^6
				q^3		

First talk

- Example.
- A key tool – cluster localization.
- Example.

Second talk

- A key tool – dealing with frozen variables.
- More examples.
- Mixed Hodge structure.

B-matrices

Let's say we have a quiver $\tilde{Q}$ with n mutable vertices and m frozen vertices. We build an $(n+m) \times n$ matrix $\tilde{B}$ whose rows are indexed by all the vertices and whose columns are indexed by the mutable vertices, with

$$B_{ij} = \#(\text{arrows } i \to j) - \#(\text{arrows } j \to i).$$

We write B for the $n \times n$ mutable part of $\tilde{B}$ and Q for the mutable part of $\tilde{Q}$.

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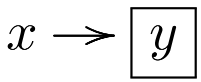

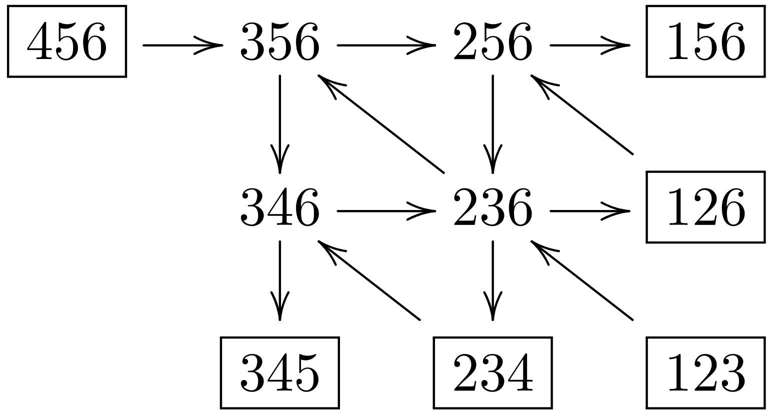

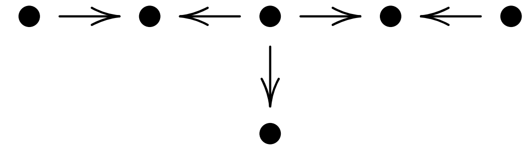

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 & -1 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

A lemma to deal with frozen variables

Lemma (Lam-S.): Let $\tilde{B}$ correspond to a cluster variety $\mathcal{X}$. Make a new $\tilde{B}$-matrix $\tilde{B}'$ by adding one more frozen row which is in the $\mathbb{Z}$-span of the rows of $\tilde{B}$, and let $\mathcal{X}'$ be the corresponding cluster variety. Then $\mathcal{X}' \cong \mathcal{X} \times \mathbb{C}^*$.

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Proof sketch: Let y be the new cluster variable of $\mathcal{X}'$. Then $\mathcal{X} \cong \{y = 1\} \subset \mathcal{X}'$. Using the linear relation between the rows of $\tilde{B}'$, we get an action of $\mathbb{C}^*$ on $\mathcal{X}'$ for which each cluster variable is rescaled by some power of $t \in \mathbb{C}^*$, and the y-variable is rescaled by t. So $\{y = 1\}$ is a slice to this action.

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Define two algebraic varieties $\mathcal{X}$ and $\mathcal{Y}$ to be ***torus equivalent*** if there is some N such that $\mathcal{X} \times (\mathbb{C}^*)^N \cong \mathcal{Y} \times (\mathbb{C}^*)^N$. Torus equivalent varieties have the same cohomology groups (Kunneth), same mixed Hodge structures and (abusing language slightly) same point counts over finite fields.

Corollary The torus equivalence class of a cluster variety only depends on the mutable part B, on the number of frozen variables, and on the $\mathbb{Z}$-span of the rows of $\tilde{B}$.

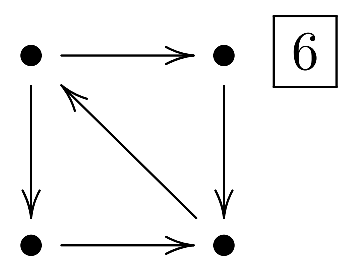
A lemma to deal with frozen variables

Corollary The torus equivalence class of a cluster variety only depends on the mutable part B, on the number of frozen variables, and on the $\mathbb{Z}$-span of the rows of $\tilde{B}$.

So, the $G(3,6)$ example on the previous slide could be shortened to "$\begin{bmatrix} 0 & -1 & -1 & 1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & 1 & 0 \end{bmatrix}$, six frozen rows, $\mathbb{Z}^4$."

Definition We'll say that a $\tilde{B}$ matrix has ***full rank*** if it has rank n, so the rows span an n-dimensional lattice. We'll say that $\tilde{B}$ has ***really full rank*** if the $\mathbb{Z}$-span of the rows is $\mathbb{Z}^n$. Both of these properties are mutation invariants. We'll also apply these words to the cluster algebra and the cluster variety.

When working with really full rank examples, I'll usually only draw the mutable part of the quiver and then put the number of frozen variables in a box, so I would represent $G(3,6)$ as



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Here is our first theorem to emphasize the usefulness of these concepts:

Theorem (Muller) A locally acyclic cluster variety of full rank is smooth.

More examples

Our last example from the previous talk was $x_1 \longrightarrow x_2$. It has $(H^0, H^1, H^2) = (\mathbb{Z}, 0, \mathbb{Z})$ and has $q^2 + 1$ points over $\mathbb{F}_q$.

Let's look at $x_1 \longrightarrow x_2 \longrightarrow x_3$. The mutable part of the B-matrix is $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$, which only has rank 2, so we need to add a frozen row if we want this to be full rank. Let's look at $x_1 \longrightarrow x_2 \longrightarrow x_3 \ \boxed{1}$.

The edge $x_2 \longrightarrow x_3$ is separating. So the cluster variety $\mathcal{X}(x_1 \longrightarrow x_2 \longrightarrow x_3 \; \boxed{1})$ is covered by $\mathcal{X}(x_1 \longrightarrow x_2 \; \boxed{2})$ and $\mathcal{X}(x_1 \; x_3 \; \boxed{2})$ with overlap $\mathcal{X}(x_1 \; \boxed{3})$.

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The number of points over $\mathbb{F}_q$ is

$$(q^2+1)(q-1)^2+(q^2-q+1)^2-(q^2-q+1)(q-1)^2 = q^4-q^3+q^2-q+1.$$

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The Mayer-Vietoris sequence is

$$\begin{array}{cccccc}
0 \longrightarrow & H^0(\mathcal{X}) & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z} \\
& H^1(\mathcal{X}) & \longrightarrow & \mathbb{Z}^2\oplus\mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^3 \\
& H^2(\mathcal{X}) & \longrightarrow & \mathbb{Z}^2\oplus\mathbb{Z}^3 & \longrightarrow & \mathbb{Z}^4 \\
& H^3(\mathcal{X}) & \longrightarrow & \mathbb{Z}^2\oplus\mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^3 \\
& H^4(\mathcal{X}) & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z}
\end{array}$$

We conclude $(H^0, H^1, H^2, H^3, H^4) = (\mathbb{Z}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z})$.

In general, we can inductively check the following result*: Consider the chain $\bullet - \bullet - \bullet - \cdots - \bullet \boxed{\text{m}}$ with n nodes. If n is odd, then assume $m \geq 1$.

The number of points over $\mathbb{F}_q$ is

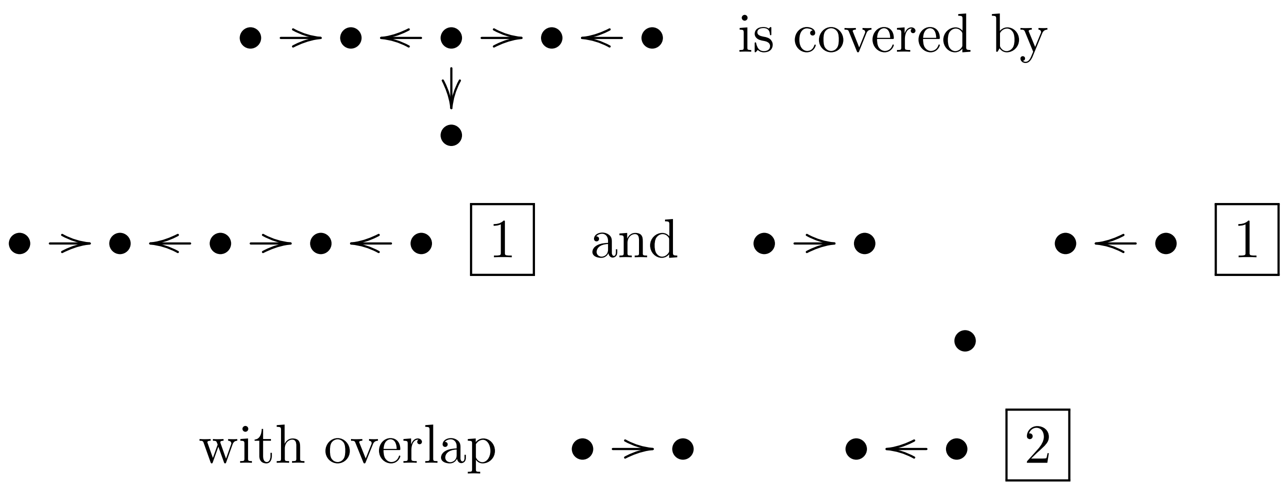
$$\begin{array}{ll} (q^n + q^{n-2} + \cdots + q^2 + 1)(q-1)^m & n \text{ even} \\ (q^n - q^{n-1} + q^{n-2} + \cdots - q^2 + q - 1)(q-1)^{m-1} & n \text{ odd} \end{array}$$

The dimension of H^j is $\pm$ the coefficient of q^j.

All boundary maps in Mayer-Vietoris are 0.

* Found earlier by Chapoton, *On the number of points over finite fields on varieties related to cluster algebras.*

But not every example is so nice. Let us turn to the E_6 diagram, which is the cluster type of $G(3,7)$.



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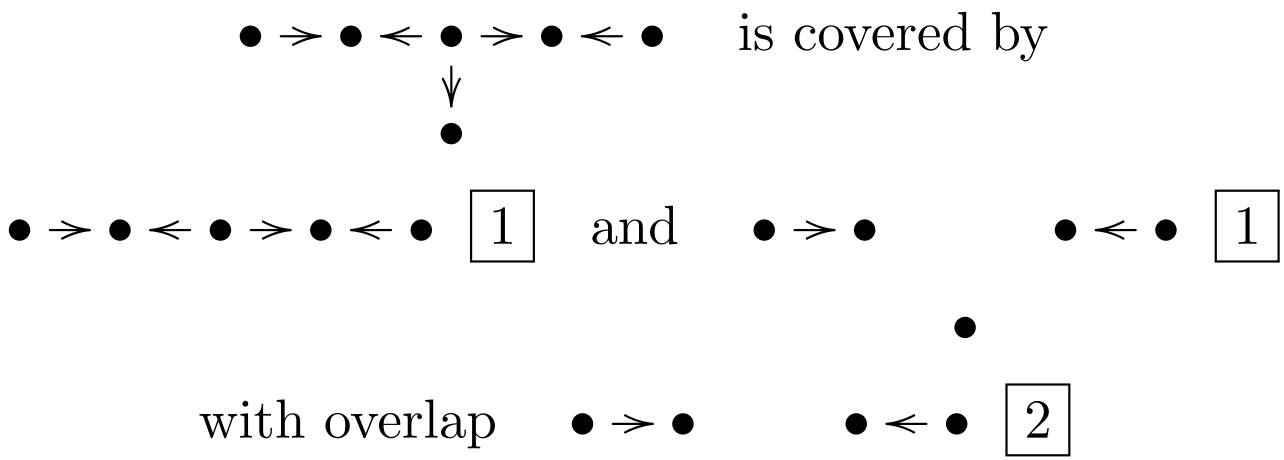

The point count is

$$\begin{aligned}
&\left(1 - q + q^2 - q^3 + q^4 - q^5 + q^6\right)\\
+&\left(1 - q + 3q^2 - 2q^3 + 3q^4 - q^5 + q^6\right)\\
-&\left(1 - 2q + 3q^2 - 4q^3 + 3q^4 - 2q^5 + q^6\right)\\
=&\ 1 + q^2 + q^3 + q^4 + q^6
\end{aligned}$$

$$
\begin{array}{rl}
 & (1 - q + q^2 - q^3 + q^4 - q^5 + q^6) \\
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= & 1 + q^2 + q^3 + q^4 + q^6
\end{array}
$$

The cohomomology is $(\mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}^2, 0, \mathbb{Z})$. Why is $H^4 \cong \mathbb{Z}^2$?

$$
\begin{array}{lllllll}
0 \longrightarrow & H^0(\mathcal{X}) & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} & \\
\to & H^1(\mathcal{X}) & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z}^2 & \\
\to & H^2(\mathcal{X}) & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z}^3 & \longrightarrow & \mathbb{Z}^3 & \\
\to & H^3(\mathcal{X}) & \longrightarrow & \mathbb{Z}^1 \oplus \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^4 & \\
\to & H^4(\mathcal{X}) & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z}^3 & \longrightarrow & \mathbb{Z}^3 & \\
\to & H^5(\mathcal{X}) & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z}^2 & \\
\to & H^6(\mathcal{X}) & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} &
\end{array}
$$

$$
\begin{aligned}
&(1 - q + q^2 - q^3 + q^4 - q^5 + q^6) \\
&+ (1 - q + 3q^2 - 2q^3 + 3q^4 - q^5 + q^6) \\
&- (1 - 2q + 3q^2 - 4q^3 + 3q^4 - 2q^5 + q^6) \\
&= 1 + q^2 + q^3 + q^4 + q^6
\end{aligned}
$$

The cohomomology is $(\mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}^2, 0, \mathbb{Z})$. Why is $H^4 \cong \mathbb{Z}^2$?

Half of it is the cokernel of $\mathbb{Z} \oplus \mathbb{Z}^2 \longrightarrow \mathbb{Z}^4$, the other half is the kernel of $\mathbb{Z} \oplus \mathbb{Z}^3 \longrightarrow \mathbb{Z}^3$.

Point counts and betti numbers are two shadows of a more complicated structure: The mixed Hodge structure.

Mixed Hodge structures

Let X be a complex algebraic variety. Then $H^k(X,\mathbb{C})$ comes equipped with a splitting, called the ***Deligne splitting***

$$H^k(X,\mathbb{C}) = \bigoplus_{p=0}^{k} \bigoplus_{q=0}^{k} H^{k,(p,q)}(X).$$

This splitting is respected by maps (1) induced functorially by maps of spaces (2) the boundary maps in Mayer-Vietores sequences (3) up to a small correction term, the maps in Gysin sequences.

Mixed Hodge structures

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Theorem: (Lam-S.) Suppose that $\mathcal{X}$ is a Louise cluster variety of full rank. Then $H^{k,(p,q)} = 0$ for $p \neq q$. Each Deligne summand $H^{k,(w,w)}(X) \subset H^k(X,\mathbb{C})$ has a basis in $H^k(X,\mathbb{Q})$.

Theorem: (Lam-S.) If $\mathcal{X}$ is Louise and really full rank, then

$$\#\mathcal{X}(\mathbb{F}_q) = \sum_{k=0}^{\dim \mathcal{X}} (-1)^{\dim X - k} \sum_{w=0}^{k} q^w \dim H^{k,(w,w)}(\mathcal{X}).$$

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If $\mathcal{X}$ is full rank, we have a formula involving Dirichlet characters. For example, $\{xx' = y^3 + 1,\ y \neq 0\}$ has cohomology $(\mathbb{Z}, \mathbb{Z}, \mathbb{Z}^3)$ and has $1 - q + \left(q^2 + q + \left(\frac{-3}{q}\right) q\right)$ points over $\mathbb{F}_q$.

Our general understanding of $H^{k,(w,w)}$ is very poor, and we are very interested in improving it. Here are the two things we do know:

Theorem (Lam-S.) We have an explicit basis for $H^{k,(k,k)}$. The basis elements are cup products of elements from $H^{1,(1,1)}$ and $H^{2,(2,2)}$.

Theorem (Lam-S.) For any fixed value of s, the polynomial

$$\sum_{w=0}^{\dim \mathcal{X}} q^w \dim H^{w+s,(w,w)}(\mathcal{X})$$

as palindromic, with center at $q^{\dim \mathcal{X}/2}$.