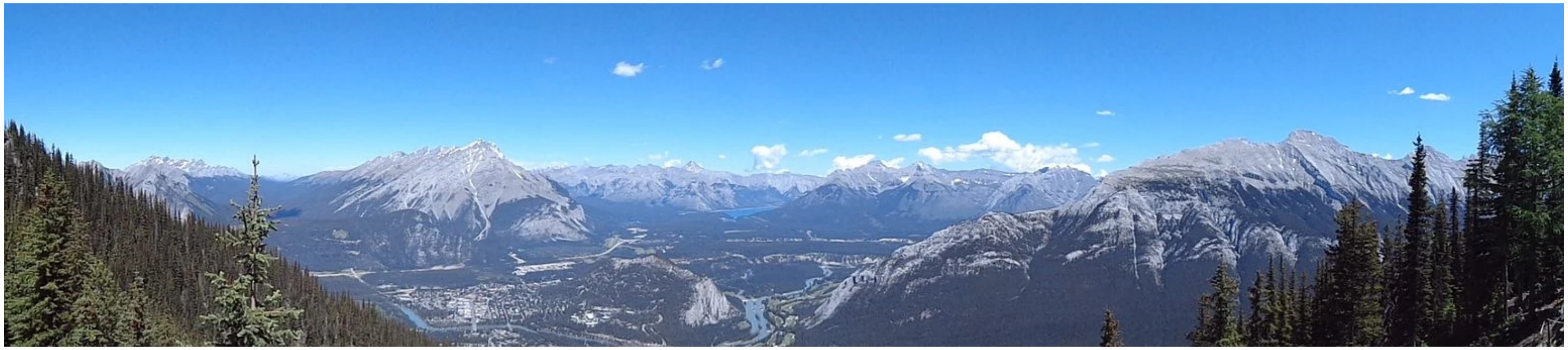
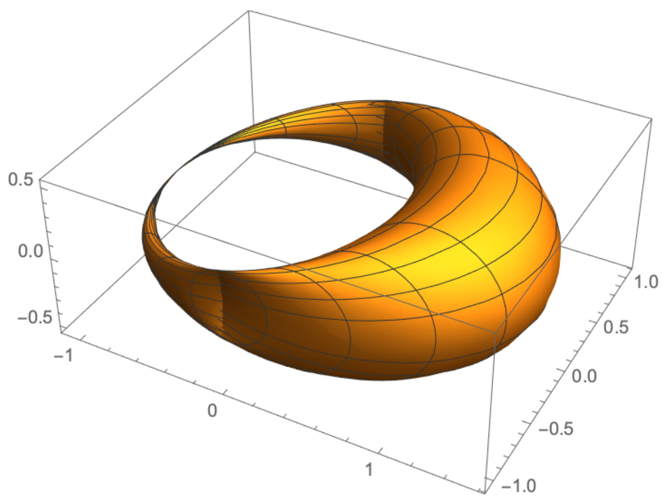
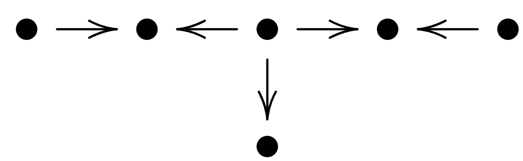


Geometry of cluster varieties I

David E Speyer





H^0	H^1	H^2	H^3	H^4	H^5	H^6
1		q^2		q^4		q^6
				q^3		

First talk

- Example.
- A key tool – cluster localization.
- Example.

Second talk

- A key tool – dealing with frozen variables.
- More examples.
- Mixed Hodge structure.

Initial example:

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$$x \longrightarrow \boxed{y}$$

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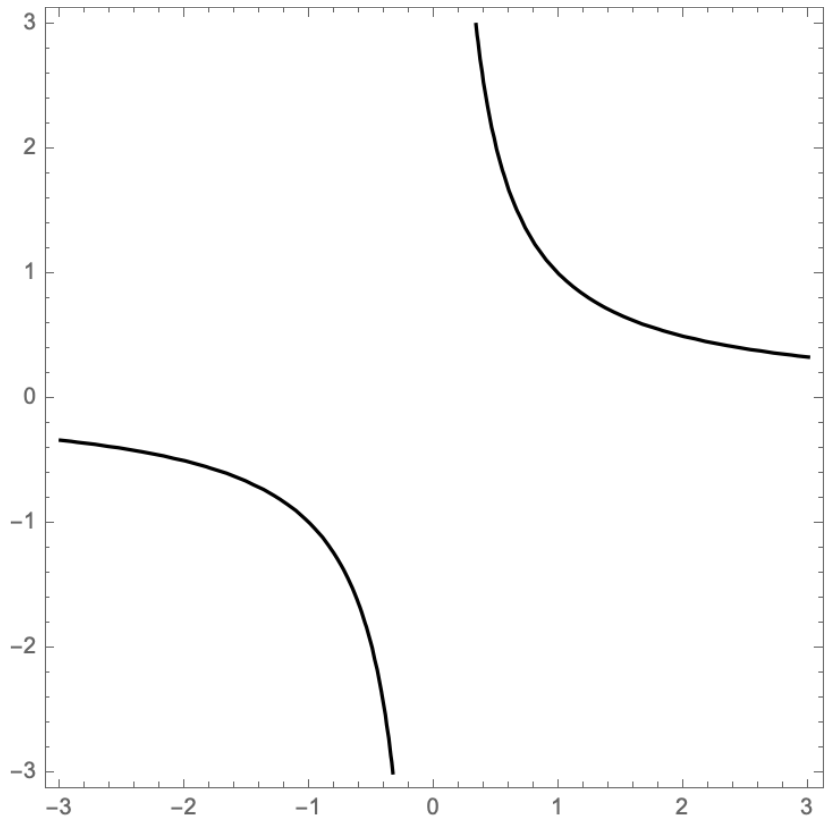
Cluster variety

$$\mathcal{Y} = \operatorname{Spec} A = \Big\{(x, x', y) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^* : xx' = y + 1\Big\}.$$

What does this look like?

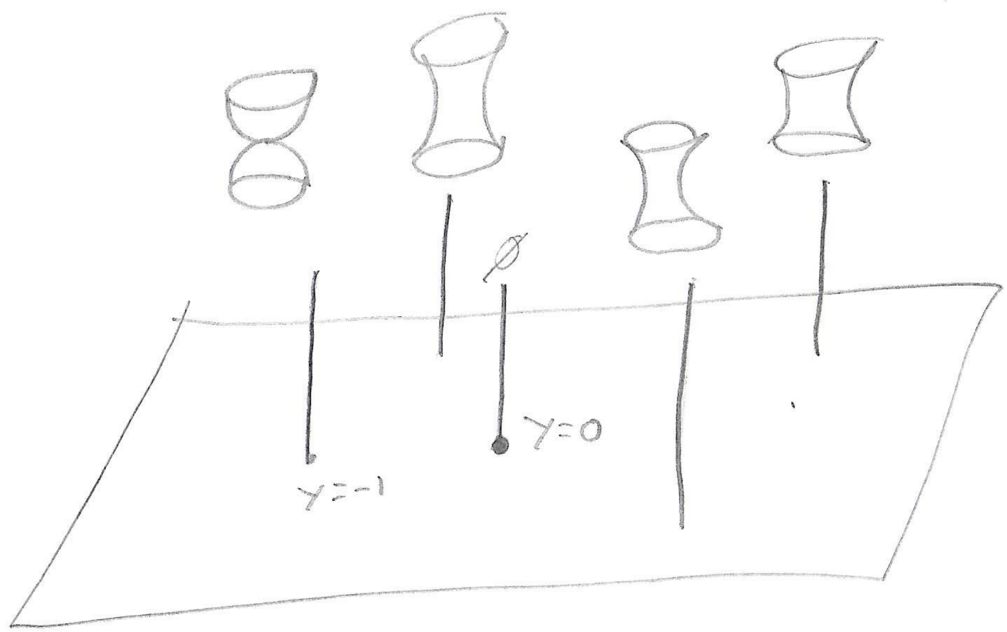
$$\left\{(x, x', y) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^* : xx' = y + 1\right\}$$

One visualization: This is $\{(x, x') \in \mathbb{C}^2 : xx' - 1 \neq 0\}$:



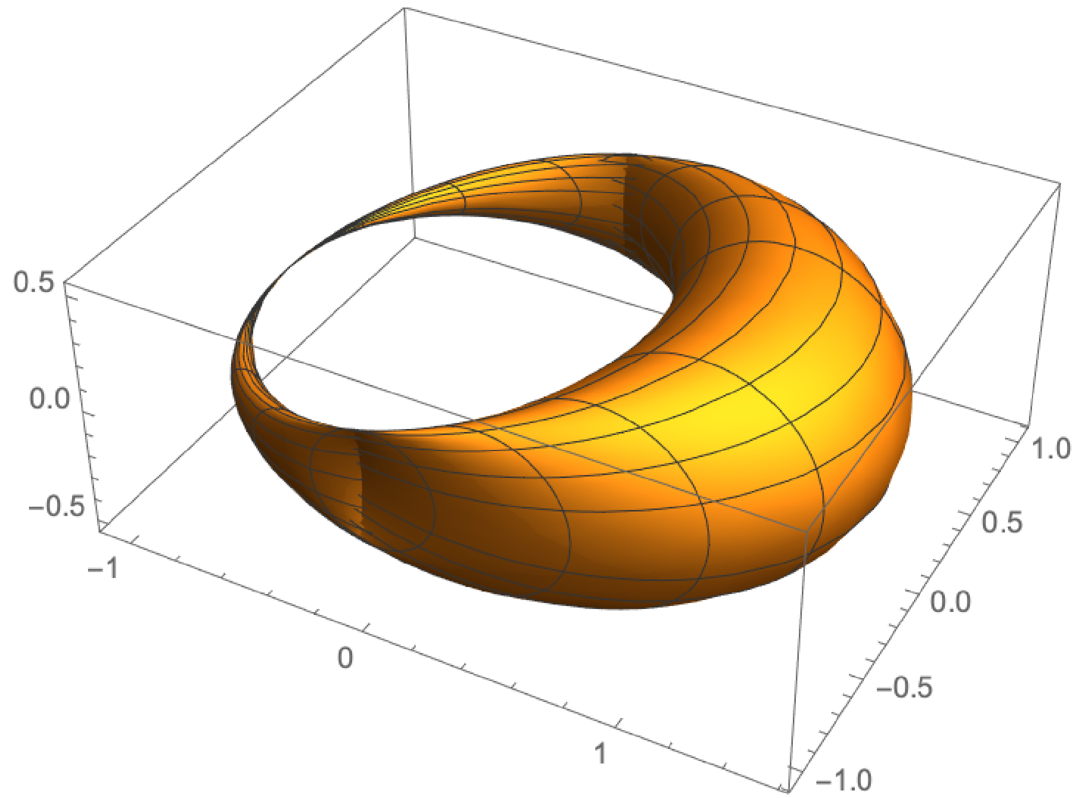
$$\Big\{(x, x', y) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^* : xx' = y + 1\Big\}$$

Another visualization: Consider the projection onto the y-coordinate. For $y \neq -1$, the fiber is a cylinder; for $y = 1$, the fiber is a pinched cylinder. Of course, there is no fiber over $y = 0$.



This has a deformation retract onto the subset $|x| = |x'|$, $|y| = 1$.

$$\left\{(x, x', y) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^* : xx' = y + 1,\ |x| = |x'|,\ |y| = 1\right\}$$



Cohomology: $H^0 = \mathbb{Z},\ H^1 = \mathbb{Z},\ H^2 = \mathbb{Z}.$

How does the cluster structure come in?

$$A = \mathbb{C}[x, x', y^{\pm 1}]/(xx' - y - 1)$$

Clusters are (x, y) and (x', y). Laurent phenomenon gives:

$$A \subset \mathbb{C}[x^{\pm 1}, y^{\pm 1}] \qquad A \subset \mathbb{C}[(x')^{\pm 1}, y^{\pm 1}].$$

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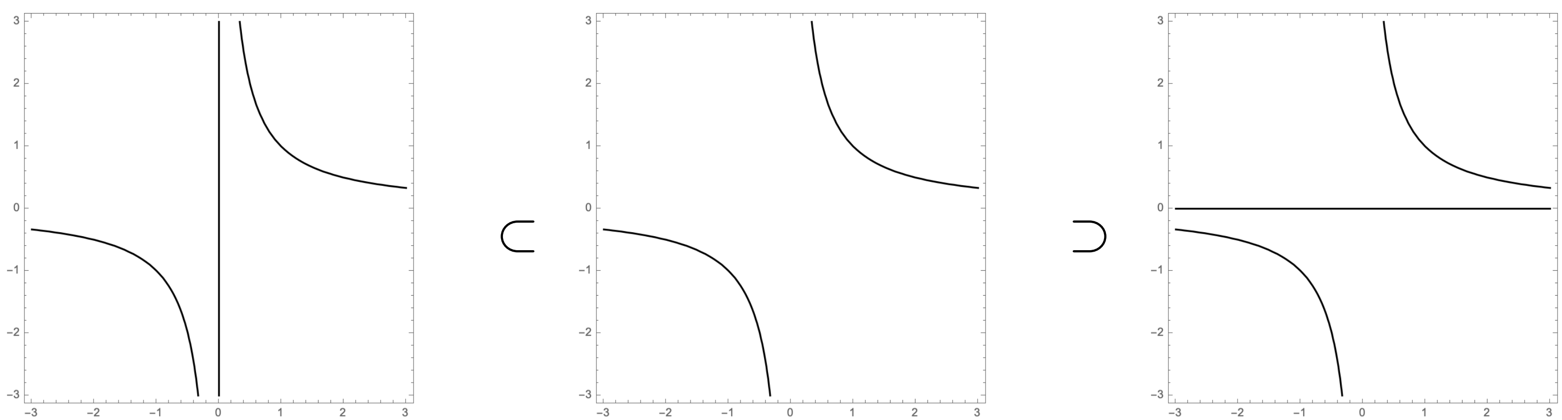
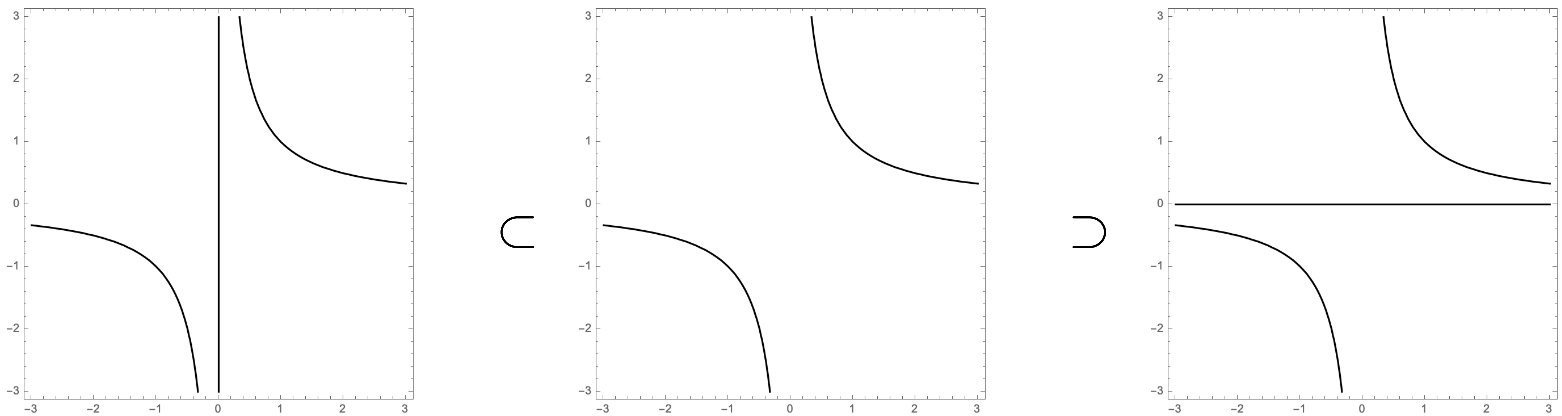
Geometrically, we have two open inclusions $(\mathbb{C}^*)^2 \to \mathcal{Y}$: One by $(x, y) \mapsto (x, \frac{1+y}{x}, y)$ and the other by $(x', y) \mapsto (\frac{1+y}{x'}, x', y)$.



Either one of these inclusions gives a decomposition $\mathcal{Y} = (\mathbb{C}^*)^2 \sqcup \mathbb{C}$.

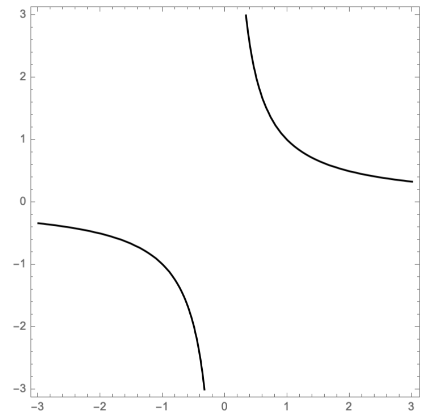


Note that the point $(0, 0, -1)$ is in neither torus.

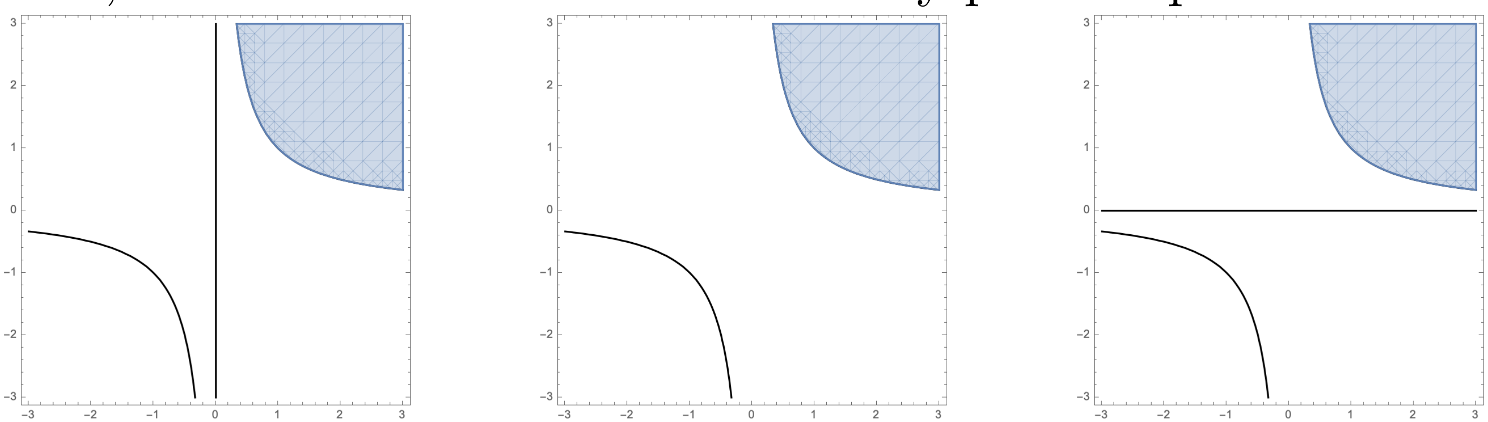
The union of cluster tori is sometimes called the ***cluster manifold***. It is messier than the affine variety $\operatorname{Spec} A$ because it is missing lots of low dimensional strata. We won't discuss it here.

Other fields:

Over $\mathbb{F}_q$, we have $q^2 - (q-1) = q^2 - q + 1$ points.



Over $\mathbb{R}$, it is natural to look at the totally positive points:



Now, onto generalities

A a cluster algebra over $\mathbb{C}$. Recall that this means that A has certain elements called ***cluster variables***, which are organized into sets called ***clusters***.

Our cluster will have size $d = n + m$, where m of the variables are ***frozen*** and in every cluster. We include the reciprocals of the frozen variables in A.

We write $\mathcal{X} = \operatorname{Spec} A$. To be precise, we are considering the complex points of $\operatorname{Spec} A$, topologized using the classical topology on $\mathbb{C}$.

For each cluster $(x_1, \ldots, x_d)$, we have the Laurent phenomenon $A \subseteq \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$. This gives a map from $\operatorname{Spec} \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}] = (\mathbb{C}^*)^d$ to $\mathcal{X}$.

I claim that this is an open inclusion. In other words, I claim that $A[(x_1 x_2 \cdots x_d)^{-1}] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$.

Proof: We have $\mathbb{C}[x_1, x_2, \ldots, x_d] \subseteq A \subseteq \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$. So $A[(x_1 x_2 \cdots x_d)^{-1}] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$. $\square$

Thus, geometrically, the open locus $\{x_1 x_2 \cdots x_d \neq 0\}$ in $\mathcal{X}$ is isomorphic to $(\mathbb{C}^*)^d$. **Each cluster gives an open torus in the cluster variety**.

We would like to make larger open sets by setting some, but not all, the variables in a cluster to be nonzero. In particular, we'd like to get open covers this way.

Reminder of basic algebraic geometry: Let A be a ring, $\mathcal{X} = \operatorname{Spec} A$, and u_1, u_2, ..., u_k functions in A. Then $\{u_j \neq 0\}$ is $\operatorname{Spec} A[u_j^{-1}]$. The open sets $\{u_j \neq 0\}$ cover $\mathcal{X}$ if and only if $(u_1, u_2, \ldots, u_k)$ generate the unit ideal.

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Let A be a cluster algebra and $\boldsymbol{x} = (x_1, x_2, \ldots, x_d)$ a cluster. Let $S \subseteq \boldsymbol{x}$ and let $x_S = \prod_{x \in S} x$. So we'd like to understand $A[x_S^{-1}]$. Is it a cluster algebra? Can we cover $\mathcal{X}$ by simpler cluster varieties of this form?

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There is a natural candidate cluster algebra: Take the quiver for $(x_1, \ldots, x_d)$ and declare the vertices x_i, for $i \in S$, to be frozen. Let $A_{\boldsymbol{x},S}$ be the resulting cluster algebra. We have $A_{\boldsymbol{x},S} \subseteq A[x_S^{-1}]$. Very often, we have equality.

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In order to think about this, we introduce the upper cluster algebra:

$$U = \bigcap_{(x_1,\ldots,x_d) \text{ a cluster}} \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}].$$

The Laurent phenomenon says that $A \subseteq U$. Very often, $A = U$.

We can define $U_{\boldsymbol{x},S}$ similarly to $A_{\boldsymbol{x},S}$. We have

$$A_{\boldsymbol{x},S} \subseteq A[x_S^{-1}] \subseteq U[x_S^{-1}] \subseteq U_{\boldsymbol{x},S}.$$

Thus, if $A_{\boldsymbol{x},S} = U_{\boldsymbol{x},S}$, then all are equal.

$$A_{\boldsymbol{x},S} \subseteq A[x_S^{-1}] \subseteq U[x_S^{-1}] \subseteq U_{\boldsymbol{x},S}$$

Theorem (Berenstein-Fomin-Zelevinsky, *Cluster Algebras III*) If there is a seed where the mutable part of the quiver is acyclic*, then $A = U$.

Corollary If the restriction of Q to the vertices not in S is acyclic, then $A[x_S^{-1}]$ is the cluster algebra $A_{\boldsymbol{x},S}$.

* A quiver is called ***acyclic*** if it has no directed cycle.

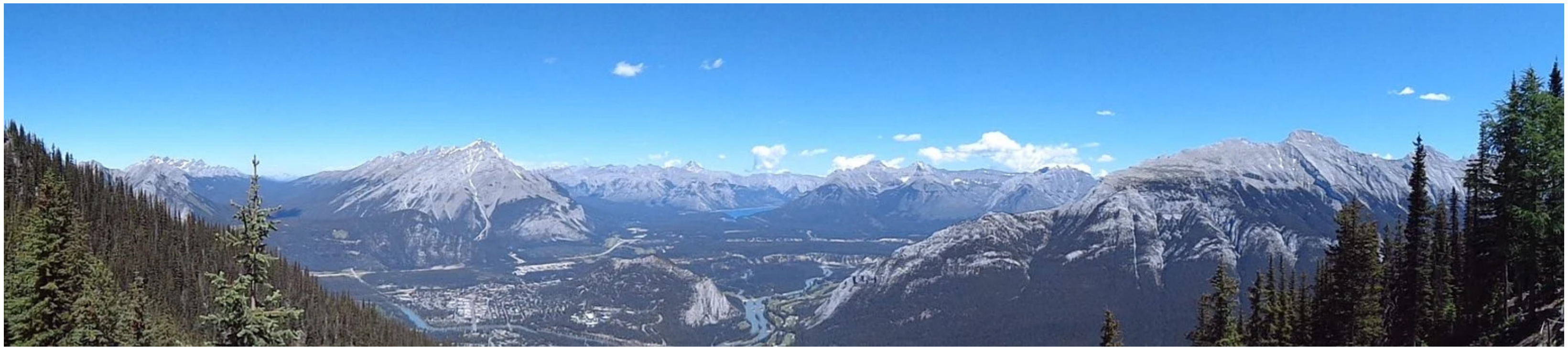
Theorem (Muller) Suppose that we have clusters $\boldsymbol{x}^1$, $\boldsymbol{x}^2$, ..., $\boldsymbol{x}^r$ and subsets S^1, S^2, ..., S^r with $A_{\boldsymbol{x}^i,S^i} = U_{\boldsymbol{x}^i,S^i}$. Suppose that the open sets $\operatorname{Spec} A_{\boldsymbol{x}^i,S^i}$ cover $\operatorname{Spec} A$. Then $A = U$.

Definition (Muller) We call A ***locally acyclic*** if $\operatorname{Spec} A$ is covered by finitely many $\operatorname{Spec} A_{\boldsymbol{x}^i,S^i}$ where the mutable part of the quiver for each $A_{\boldsymbol{x}^i,S^i}$ is acyclic.

Definition (Muller) Let Q be a quiver. We define an edge $i \to j$ to be a ***separating edge*** of Q if there is no bi-infinite directed walk containing $i \to j$.

Theorem (Muller) If $i \to j$ is a separating edge, then
$\operatorname{Spec} A = \operatorname{Spec} A[x_i^{-1}] \cup \operatorname{Spec} A[x_j^{-1}]$.

These results are from *$A = U$ for Locally Acyclic Cluster Algebras*. See also Muller's earlier paper *Locally Acyclic Cluster Algebras*.



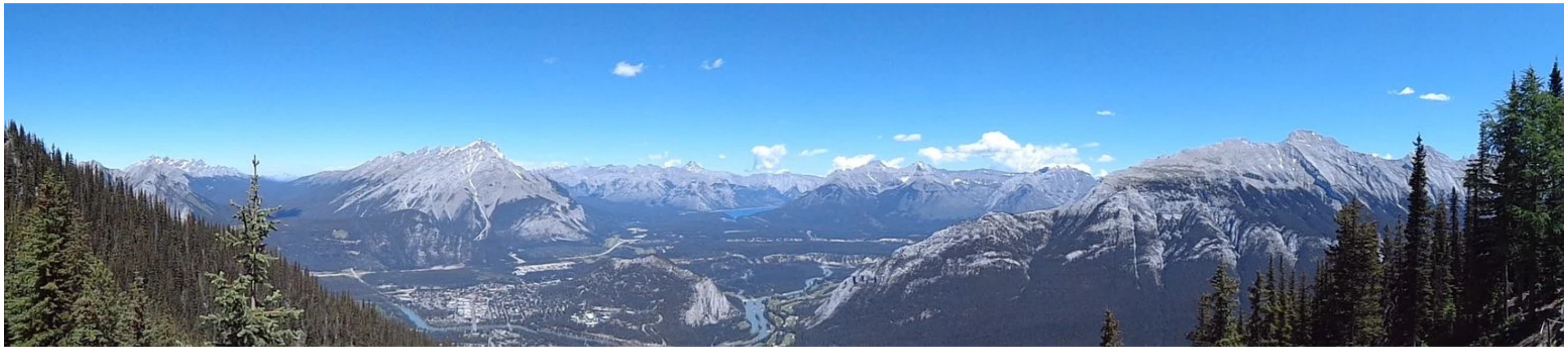
The Banff algorithm

Step 0: If Q has no edges, halt. In this case, $\mathcal{X}$ looks a lot like our original example $\mathcal{Y}$; see Lam-S., *Cohomology of cluster varieties. I. Locally acyclic case*, Section 7.

Step 1: If Q has edges, mutate Q until you find a quiver Q' with a separating edge $i \to j$. In this case, you know that $\operatorname{Spec} A = \operatorname{Spec} A[x_i^{-1}] \cup \operatorname{Spec} A[x_j^{-1}]$.

Step 2: Recurse on the quivers $Q' \setminus \{i\}$ and $Q' \setminus \{j\}$.

If every branch of this recursion halts, then every localization that you met along the way was a cluster algebra and you will have computed an open cover of $\mathcal{X}$ by simple spaces.



Define a cluster algebra to be ***Banff*** if, either Q has no edges, or else Q has a separating edge $x_i \to x_j$, and $A[x_i^{-1}]$ and $A[x_j^{-1}]$ are both Banff. In other words, the Banff algorithm stops.

Define a cluster algebra to be ***Louise*** if, either Q has no edges, or else Q has a separating edge $x_i \to x_j$, and $A[x_i^{-1}]$, $A[x_j^{-1}]$ and $A[(x_i x_j)^{-1}]$ are all Louise.

A final example

$$x_1 \longrightarrow x_2.$$

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This is the cluster algebra of type A_2 with no frozen variables.

It is $\{\Delta_{12} = \Delta_{23} = \Delta_{34} = \Delta_{45} = \Delta_{15} = 1\}$ inside $G(2,5)$.

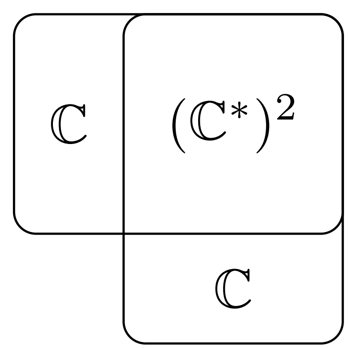
It is also $\{(\Delta_{ij}) \in G(2,5) : \Delta_{12}\Delta_{23}\Delta_{34}\Delta_{45}\Delta_{15} \neq 0\}/(\mathbb{C}^*)^5$.

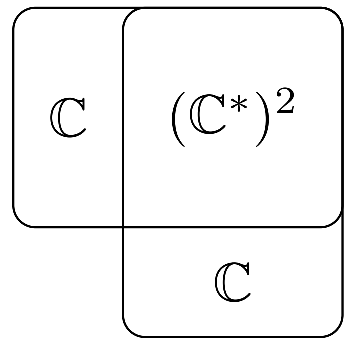
A final example

$$x_1 \longrightarrow x_2.$$

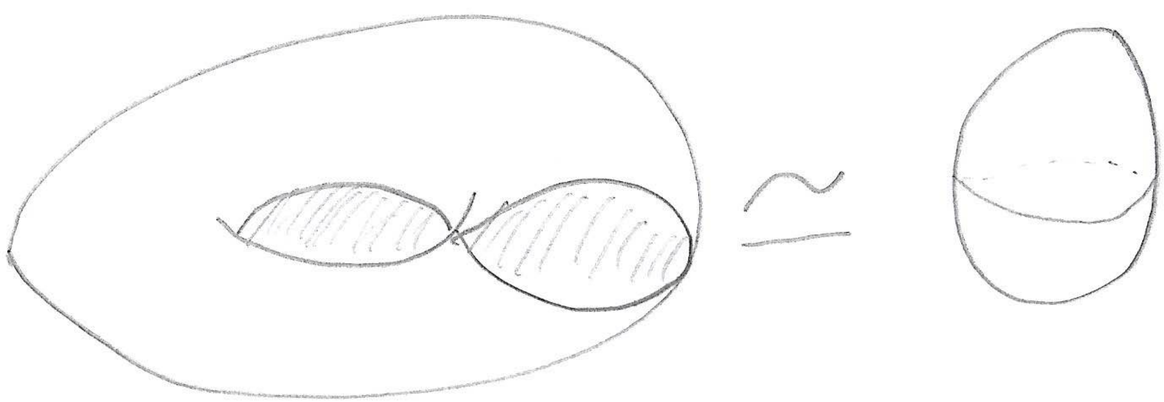
The edge is separating, so $\mathcal{X}$ is covered by the localizations $\mathcal{Y}_1 := \{x_1 \neq 0\}$ and $\mathcal{Y}_2 := \{x_2 \neq 0\}$. These have quivers $\boxed{x_1} \longrightarrow x_2$ and $x_1 \longrightarrow \boxed{x_2}$. Each localization is isomorphic to our original example, $\mathcal{Y}$.

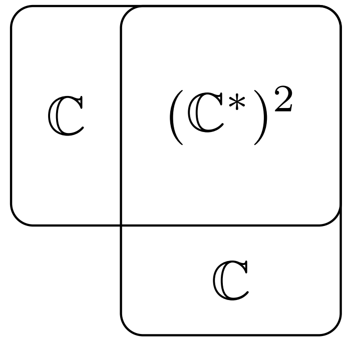
The intersection $\mathcal{Y}_1 \cap \mathcal{Y}_2$ is $\{x_1 x_2 \neq 0\}$. This is the torus $(\mathbb{C}^*)^2$.





Topologically, this is homotopic to $(S^1)^2$ with two discs glued in, bounding circles in the two directions – in other words, S^2.



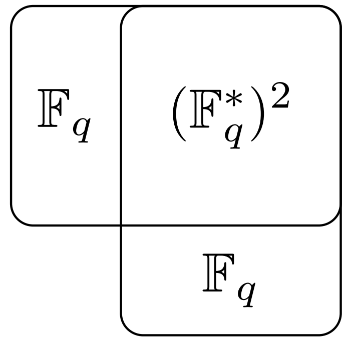


In terms of cohomology, we have a Meyer-Vietores sequence

$$
\begin{array}{ccccccc}
0 & \longrightarrow & H^0(\mathcal{X}) & \longrightarrow & H^0(\mathcal{Y}_1) \oplus H^0(\mathcal{Y}_2) & \longrightarrow & H^0(\mathcal{Y}_1 \cap \mathcal{Y}_2) \\
 & \longrightarrow & H^1(\mathcal{X}) & \longrightarrow & H^1(\mathcal{Y}_1) \oplus H^1(\mathcal{Y}_2) & \longrightarrow & H^1(\mathcal{Y}_1 \cap \mathcal{Y}_2) \\
 & \longrightarrow & H^2(\mathcal{X}) & \longrightarrow & H^2(\mathcal{Y}_1) \oplus H^2(\mathcal{Y}_2) & \longrightarrow & H^2(\mathcal{Y}_1 \cap \mathcal{Y}_2)
\end{array}
$$

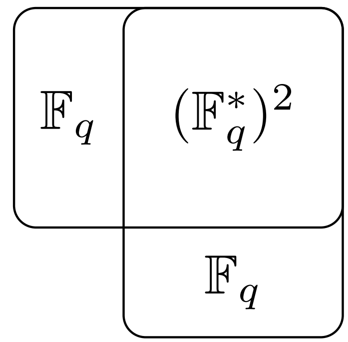
$$
\begin{array}{ccccccc}
0 & \longrightarrow & H^0(\mathcal{X}) & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} \\
 & \longrightarrow & H^1(\mathcal{X}) & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z}^2 \\
 & \longrightarrow & H^2(\mathcal{X}) & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z}
\end{array}
$$

So $H^0(\mathcal{X}) = \mathbb{Z}$, $H^1(\mathcal{X}) = 0$, $H^2(\mathcal{X}) = \mathbb{Z}$.



In terms of counting points over $\mathbb{F}_q$, we have

$$\#\mathcal{X}(\mathbb{F}_q) = (q-1)^2 + q + q = q^2 + 1.$$



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A teaser for a recent paper: This is a q-Catalan number! See Galashin and Lam *Positroids, knots, and* (q,t)*-Catalan numbers* for much more.