# Cluster structures in commutative rings

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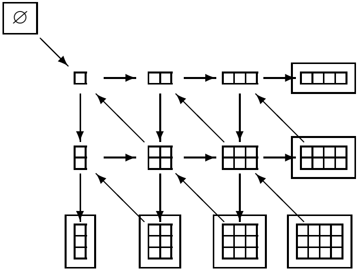
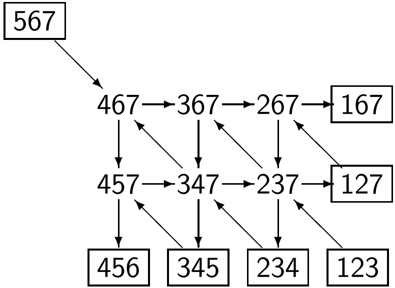




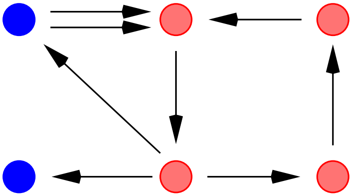
# Outline of my talks

Talk 1: What is a cluster algebra?

Talk II: Cluster structures in commutative rings

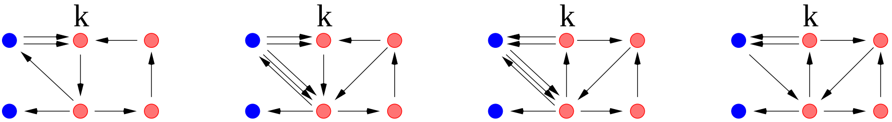
- How can we identify a commutative ring with a cluster algebra?
- Starfish lemma
- The Grassmannian, revisited
- Defining cluster algebras by generators and relations
- Content mostly based on Fomin-Williams-Zelevinsky, Introduction to Cluster Algebras, Chapter 6, *arXiv:2008.09189.*

# Quivers



A *quiver* is a finite directed graph.
Two types of vertices: "frozen" and "mutable."

# Quiver Mutation




Let $k$ be a mutable vertex of $Q$.

## Quiver mutation $\mu_k : Q \mapsto Q'$ is computed in 3 steps:

1. For each instance of $j \to k \to \ell$, introduce an edge $j \to \ell$.
2. Reverse the direction of all edges incident to $k$.
3. Remove oriented 2-cycles.

# Seeds

Let $\mathcal{F}$ be a field of rational functions in $m$ independent variables over $\mathbb{C}$. A *seed* in $\mathcal{F}$ is a pair $(Q, \mathrm{x})$ consisting of:

- a quiver $Q$ on $m$ vertices
- an *extended cluster* x, an $m$-tuple of algebraically independent (over $\mathbb{C}$) elements of $\mathcal{F}$, indexed by the vertices of $Q$.

coefficient variables $\leftrightarrow$ frozen vertices

cluster variables $\leftrightarrow$ mutable vertices

$$\text{Cluster} = \{\text{cluster variables}\}$$

$$\text{Extended Cluster} = \{\text{cluster variables, coefficient variables}\}$$

# Seed mutation

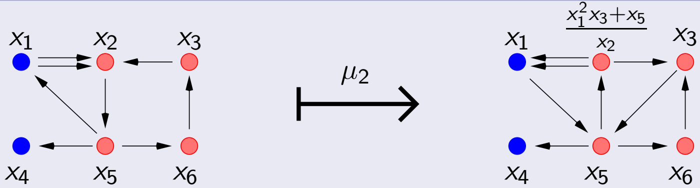
Let $k$ be a mutable vertex in $Q$ and let $x_k$ be the corresponding cluster variable. Then the seed mutation $\mu_k : (Q, \mathrm{x}) \mapsto (Q', \mathrm{x}')$ is defined by

- $Q' = \mu_k(Q)$
- $\mathrm{x}' = \mathrm{x} \cup \{x'_k\} \setminus \{x_k\}$, where

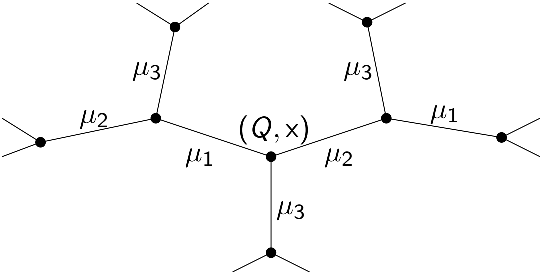
$$x_k x'_k = \prod_{j \leftarrow k} x_j + \prod_{j \rightarrow k} x_j \text{ (is the } \textit{exchange relation}\text{)}$$

Remark: Mutation is an involution.

## Example



# Definition of cluster algebra




Let $(Q, \mathrm{x})$ be a seed in $\mathcal{F}$, where $Q$ has $n$ mutable vertices.
Consider the $n$-regular tree $\mathbb{T}_n$ with vertices labeled by seeds, obtained by applying all possible sequences of mutations to $(Q, \mathrm{x})$.
Let $\chi$ be the union of all cluster variables which appear at nodes of $\mathbb{T}_n$.
Let $x_{n+1}, \dots, x_m$ denote the coefficient variables.
Let the *ground ring* be $\mathcal{R} = \mathbb{C}[x_{n+1}, \dots, x_m]$ (or $\mathbb{C}[x_{n+1}^{\pm}, \dots, x_m^{\pm}]$).
The *cluster algebra* $\mathcal{A}(Q) := \mathcal{R}[\chi] \subset \mathcal{F}$ is the $\mathcal{R}$-subalg generated by $\chi$.
We say it has *rank n*.

# Identifying coordinate rings with cluster algebras

*Proposition.* Let $X$ be rational affine irreducible algebraic variety of dim $m$. Let $\mathcal{A}$ be rank $n$ cluster algebra with cluster vars $\mathcal{X}$ and frozen variables $x_{n+1}, \ldots, x_m$. Suppose we have nonzero *regular functions* on $X$

$$\{\varphi_z : z \in \mathcal{X}\} \cup \{\varphi_{n+1}, \ldots, \varphi_m\} \subset \mathbb{C}[X] \text{ such that}$$

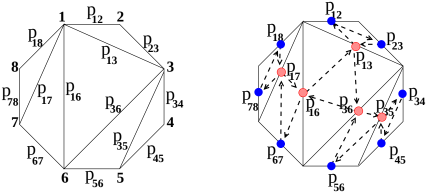
- the functions $\varphi_z$ $(z \in \mathcal{X})$ and $\varphi_i$ $(n+1 \leq i \leq m)$ generate $\mathbb{C}[X]$;
- replacing each cluster variable $z$ by $\varphi_z$, and each frozen variable $x_i$ by $\varphi_i$ makes every exchange relation into an identity in $\mathbb{C}[X]$.

Then there is a unique $\mathbb{C}$-algebra isomorphism $\varphi : \mathcal{A} \to \mathbb{C}[X]$ such that $\varphi(z) = \varphi_z$ for all $z \in \mathcal{X}$ and $\varphi(x_i) = \varphi_i$ for $i \in \{n+1, \ldots, m\}$.

- Note: Irred+rationality implies that fraction field of $\mathbb{C}[X]$ is well-defined and equals the field of rational functions over $\mathbb{C}$ in $\dim(X)$ indep variables.
- Main thing to prove is that each cluster in $\mathcal{A}$ gives rise to transcendence basis in $\mathbb{C}(X)$.
- In practice, we can only use Prop for cluster algebras of FINITE TYPE.

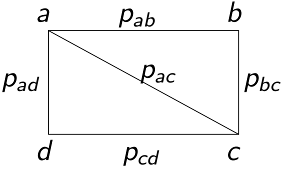
# Application of Prop to $\mathbb{C}[Gr_{2,n+3}]$

Each triangulation of an $(n+3)$-gon gives rise to an initial seed for a cluster algebra as follows:




- Identify sides/diagonals of polygon with Plücker coords for $Gr_{2,n+3}$.
- The Plücker coords are regular functions and generate $\mathbb{C}[Gr_{2,n+3}]$.
- Exchange relations in cluster algebra correspond to Plücker relations.

Exchange relation:




$$p_{ac}p_{bd} = p_{ab}p_{cd} + p_{bc}p_{ad}$$

# Starfish Lemma

Say you have a subset of elements of $\mathbb{C}[X]$ which you think forms an initial cluster for a cluster structure on $\mathbb{C}[X]$. If cluster algebra has infinitely many cluster var's, need practical way to check each one is regular!

## Proposition ("Starfish lemma")

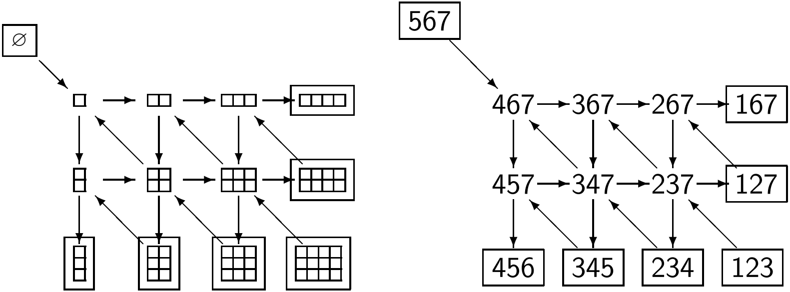
Let $\mathcal{R} = \mathbb{C}[X]$ be the coordinate ring of an irred normal affine complex algebraic variety $X$, with $\mathbb{C}(X)$ the field of rational functions. Let $(Q, \tilde{x})$ be seed of rank $n$ in $\mathbb{C}(X)$ with $\tilde{x} = (x_1, \ldots, x_m)$ for $n \leq m$ such that

1. all elements of $\tilde{x}$ belong to $\mathcal{R}$;
2. the cluster variables in $\tilde{x}$ are pairwise coprime;
3. for each cluster variable $x_k \in \tilde{x}$, the seed mutation $\mu_k$ replaces $x_k$ with an element $x_k'$ that lies in $\mathcal{R}$ and is coprime to $x_k$.

Then $\mathcal{A}(Q, \tilde{x}) \subset \mathcal{R}$.

- Proof of Starfish lemma is application of *Hartogs' principle*, which says a function on $X$ which is regular outside a subset of codim 2 is regular everywhere.

# The Grassmannian, revisited




*Exercise:* Consider the above *rectangles seed* (encoded in two ways) for $Gr_{3,7}$. Show that its generalization gives a cluster structure on $\mathbb{C}[Gr_{k,n}]$.

- Show that if one mutates at any mutable cluster variable above, the new cluster var is a *regular function* which is *coprime* to the old cluster var (so can apply Starfish Lemma).
- Show that one can obtain any Plücker coordinate from the above quiver by an appropriate sequence of mutations.

# Cluster structures on Grassmannians and generalizations

Cluster structures in Grassmannians and subvarieties:

- Fomin-Zelevinsky: $Gr_{2,n}$ (gave two non-isomorphic structures)
- Scott: $Gr_{k,n}$, using Postnikov's plabic graphs. (Proof outlined in previous exercise is different, from `arXiv:2008.09189`).
- Fomin-Pylyavskyy: multiple cluster structures on $Gr_{3,n}$, using tensor diagrams.
- Serhiyenko–Sherman-Bennett–W.: *open Schubert varieties* in $Gr_{k,n}$.
- Galashin-Lam: *open positroid varieties* in $Gr_{k,n}$. Partial progress had also appeared in Müller-Speyer.
- Proofs for the Schubert/positroid cases use Leclerc's work on cluster structures in Richardson varieties, which uses *categorification*. Would be nice to give elementary approach.
- Fraser–Sherman-Bennett: found additional (non-isomorphic) cluster structures for Schubert/positroid varieties.

# Defining cluster algebras by generators and relations

- Commutative algebras are typically described in terms of generators and relations, e.g. $\mathcal{A} = \mathbb{C}[z_1, z_2, \dots]/I$ for ideal $I \subset \mathbb{C}[z_1, z_2, \dots]$. Usually $\{z_1, z_2, \dots\}$ is finite and $I$ is finitely generated.
- In contrast, a cluster algebra $\mathcal{A}$ is defined inside a field $\mathcal{F}$ of rational functions as the algebra generated by certain (recursively determined) elements of $\mathcal{F}$, the cluster variables.
  - The set of cluster variables in a cluster algebra is typically *infinite*.
  - The relations among generators are not given explicitly but we do know some of them, the exchange relations.
  - Many cluster algebras are *not finitely generated.*
    E.g. a cluster algebra of rank 3 with no frozen variables is finitely generated iff it has an acyclic seed (Berenstein-Fomin-Zelevinsky). Generalization in setting of locally acyclic cluster algebras (Müller).

# Defining cluster algebras by generators and relations

Even when a cluster algebra has finite type, the ideal of relations satisfied by the cluster variables may not be generated by the exchange relations.

- Consider $\mathbb{C}[Gr_{3,6}]$; this is a finite type cluster algebra of type $D_4$. The set of cluster variables includes the Plücker coordinates.
- The Plücker coordinates satisfy

$$P_{135}P_{246} - P_{134}P_{256} - P_{136}P_{245} - P_{123}P_{456} = 0.$$

- However, this relation cannot be written as a polynomial combination of exchange relations, since all those relations have degree at least two, and none involve the monomial $P_{135}P_{246}$.

# Defining cluster algebras by generators and relations

Suppose a cluster algebra $\mathcal{A}$ of rank $n$ is finitely generated.
Then the cluster vars appearing in some finite subtree $T$ of $\mathbb{T}_n$ generate $\mathcal{A}$.
Let $\chi_T$ denote the set of formal variables which consists of

- one formal variable for each frozen variable
- one formal variable for each distinct cluster var appearing in $T$

Let $I_T \subset \mathbb{C}[\chi_T]$ be the *exchange ideal* generated by the exchange relations corresponding to edges of $T$.
Let $M_T$ denote the product of all mutable elements of $\chi_T$. Set

$$J_T = \{f \in \mathbb{C}[\chi_T] \mid (M_T)^a f \in I_T \text{ for some a}\}.$$

This is the *saturation of $I_T$*, the set of polys that can be multiplied by a monomial in mutable cluster var in $\chi_T$ so that the product lies in $I_T$.

**Theorem (Fomin–W.), `arXiv:2008.09189` (Theorem 6.8.10)**

We have a canonical isomorphism $\mathcal{A} \cong \mathbb{C}[\chi_T]/J_T$.

# Defining cluster algebras by generators and relations

**Exercise**

Although the equation

$$P_{135}P_{246} - P_{134}P_{256} - P_{136}P_{245} - P_{123}P_{456} = 0$$

does not lie in the ideal generated by the exchange relations, show that we can multiply it by a monomial in the Plücker coordinates so that the result lies in the exchange ideal.

# References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