Introduction to Cluster Algebras

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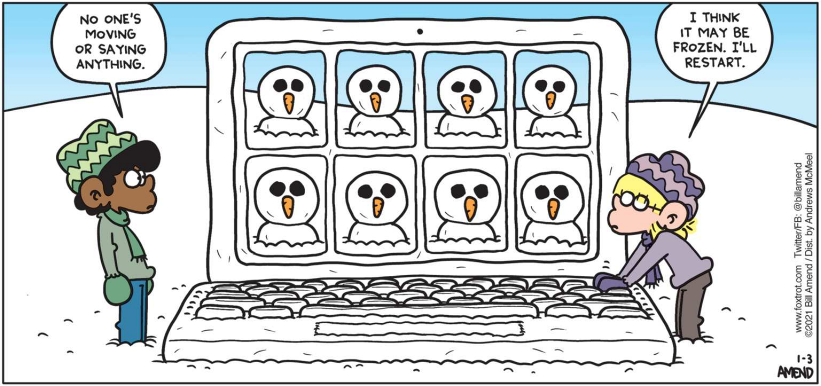


Overview

- Cluster algebras are commutative rings with distinguished generators (*cluster variables*) having a remarkable combinatorial structure.
- The structure of a cluster algebra is encoded by a quiver, and the relations among the cluster variables are encoded by *quiver mutation*.
- Cluster algebras were introduced by Fomin and Zelevinsky in 2000, motivated by total positivity and Lusztig's canonical basis.

Cluster algebras have since appeared in many other contexts such as:

- Poisson geometry
- triangulations of surfaces and Teichmüller theory
- tropical geometry
- mathematical physics: wall-crossing phenomena, quiver gauge theories, scattering amplitudes, soliton solutions to the KP equation

Outline of my talks

Talk 1: What is a cluster algebra?

- Motivation from total positivity
- Quivers and quiver mutation
- Seeds and seed mutation
- Definition of cluster algebra
- Cluster algebras in nature: surfaces, Grassmannians

Talk II: Cluster structures in commutative rings

- How can we identify a commutative ring with a cluster algebra?
- Starfish lemma
- The Grassmannian, revisited
- Presentations by generators and relations?

The Grassmannian and its positive part

The Grassmannian $Gr_{k,n}(\mathbb{R}) = \{V \mid V \subset \mathbb{R}^n, \dim V = k\}$
Represent an element of $Gr_{k,n}(\mathbb{R})$ by a full-rank $k \times n$ matrix A.

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 3 & 2 \end{pmatrix}$$

Can think of $Gr_{k,n}(\mathbb{R})$ as $Mat_{k,n}/\sim$.

Given $I \in \binom{[n]}{k}$, the *Plücker coordinate* $\Delta_I(A)$ is the minor of the $k \times k$ submatrix of A in column set I.

The *totally positive part* of the Grassmannian $(Gr_{k,n})_{>0}$ is the subset of $Gr_{k,n}(\mathbb{R})$ where all Plucker coordinates $\Delta_I(A) > 0$.

A $k \times n$ matrix A has $\binom{n}{k}$ Plücker coordinates.
How many (and which ones) do we need to test to determine whether A represents a point of $(Gr_{k,n})_{>0}$?

The Grassmannian and its positive part

Represent an element of $Gr_{k,n}(\mathbb{R})$ by a full-rank $k \times n$ matrix A.

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 3 & 2 \end{pmatrix}$$

Given $I \in \binom{[n]}{k}$, the *Plücker coordinate* $\Delta_I(A)$ is the minor of the $k \times k$ submatrix of A in column set I.

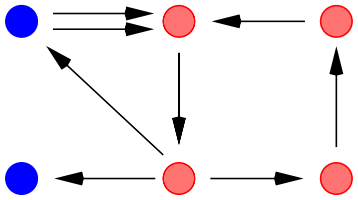
The Plücker coordinates satisfy

$$\Delta_{13}(A)\Delta_{24}(A) = \Delta_{12}(A)\Delta_{34}(A) + \Delta_{14}(A)\Delta_{23}(A).$$

So if $\Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{14}$ and Δ_{24} are positive, so is Δ_{13}.
Or if $\Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{14}$ and Δ_{13} are positive, so is Δ_{24}.

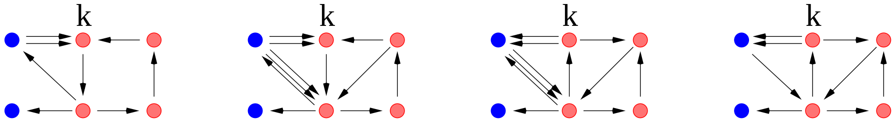
How can we generalize this picture to $Gr_{2,n}(\mathbb{R})$? $Gr_{k,n}(\mathbb{R})$?

Quivers



A *quiver* is a finite directed graph.
Multiple edges are allowed.
Oriented cycles of length 1 or 2 are forbidden.
Two types of vertices: "frozen" and "mutable."
Ignore edges connecting frozen vertices.

Quiver Mutation



Let k be a mutable vertex of Q.

Quiver mutation $\mu_k : Q \mapsto Q'$ is computed in 3 steps:

1. For each instance of $j \to k \to \ell$, introduce an edge $j \to \ell$.
2. Reverse the direction of all edges incident to k.
3. Remove oriented 2-cycles.

Mutation is an involution, i.e. $\mu_k^2(Q) = Q$ for each vertex k.

Two quivers are *mutation-equivalent* if one can get between them via a sequence of mutations. *Show aplet.*

Seeds

Let $\mathcal{F}$ be a field of rational functions in m independent variables over $\mathbb{C}$. A *seed* in $\mathcal{F}$ is a pair (Q, x) consisting of:

- a quiver Q on m vertices
- an *extended cluster* x, an m-tuple of algebraically independent (over $\mathbb{C}$) elements of $\mathcal{F}$, indexed by the vertices of Q.

coefficient variables $\leftrightarrow$ frozen vertices

cluster variables $\leftrightarrow$ mutable vertices

Cluster $=$ {cluster variables }

Extended Cluster $=$ {cluster variables, coefficient variables}

Seed mutation

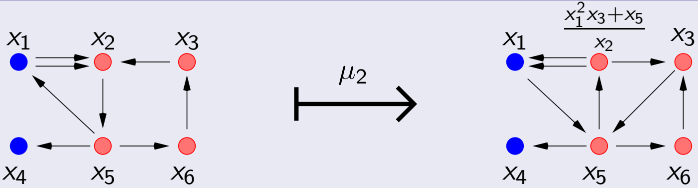
Let k be a mutable vertex in Q and let x_k be the corresponding cluster variable. Then the seed mutation $\mu_k : (Q, \mathrm{x}) \mapsto (Q', \mathrm{x}')$ is defined by

- $Q' = \mu_k(Q)$
- $\mathrm{x}' = \mathrm{x} \cup \{x_k'\} \setminus \{x_k\}$, where

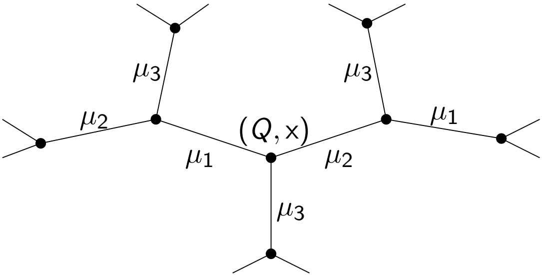
$$x_k x_k' = \prod_{j \leftarrow k} x_j + \prod_{j \to k} x_j \text{ (is the \textit{exchange relation})}$$

Remark: Mutation is an involution.

Example



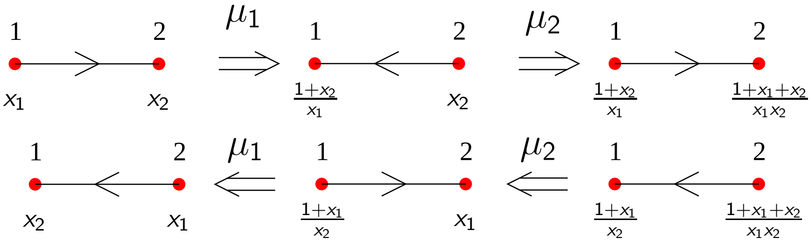
Definition of cluster algebra



Let (Q, x) be a seed in $\mathcal{F}$, where Q has n mutable vertices.
Consider the n-regular tree $\mathbb{T}$ with vertices labeled by seeds, obtained by applying all possible sequences of mutations to (Q, x).
Let χ be the union of all cluster variables which appear at nodes of $\mathbb{T}$.
Let the *ground ring* be $\mathcal{R} = \mathbb{C}[x_{n+1}, \dots, x_m]$, the polynomial ring generated by frozen variables. (Alternatively let $\mathcal{R} = \mathbb{C}[x_{n+1}^{\pm}, \dots, x_m^{\pm}]$.)
The *cluster algebra* $\mathcal{A}(Q) := \mathcal{R}[\chi] \subset \mathcal{F}$ is the $\mathcal{R}$-subalg generated by χ.

Example

Consider the following seed (Q, x), where $\mathrm{x} = \{x_1, x_2\}$.



The cluster algebra $\mathcal{A}(Q)$ is the subring of $\mathcal{F} = \mathbb{C}(x_1, x_2)$ generated by all cluster variables $\chi = \{x_1, x_2, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1x_2}, \frac{1+x_1}{x_2}.\}$

Note: every cluster variable is a Laurent polynomial in $\{x_1, x_2\}$.
Note: each Laurent polynomial has positive coefficients.
Note: there are finitely many cluster variables.
The 2-regular tree closes up to form a pentagon.

Fundamental results

Let $\mathcal{A} = \mathcal{A}(Q)$ be an arbitrary cluster algebra, with initial seed (Q, x).

Laurent phenomenon (Fomin + Zelevinsky)

Every cluster variable is a Laurent polynomial in the variables from x (the *initial cluster variables*).

Positivity Theorem (Lee-Schiffler, Gross-Hacking-Keel)

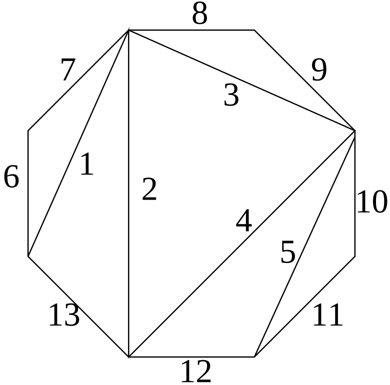
Each such Laurent polynomial has positive coefficients.

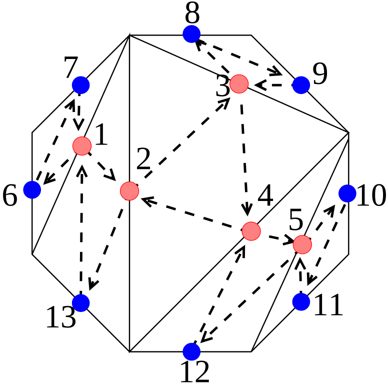
Finite type classification (Fomin + Zelevinsky)

We say $\mathcal{A}$ has *finite type* if there are only finitely many cluster variables.
The finite type cluster algebras are classified by Dynkin diagrams.
When $\mathcal{A}$ is of finite type, the n-regular tree closes up on itself and becomes the 1-skeleton of a convex polytope.

Cluster algebras and triangulations

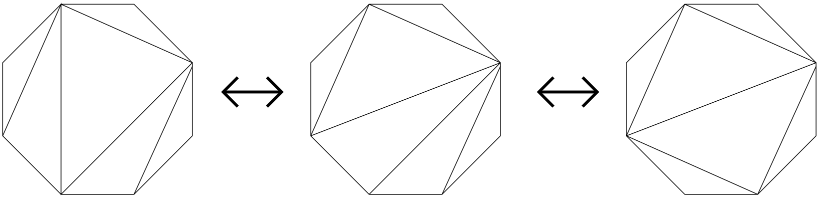
Fix a triangulation T of a d-gon. We associate to it a quiver Q_T:



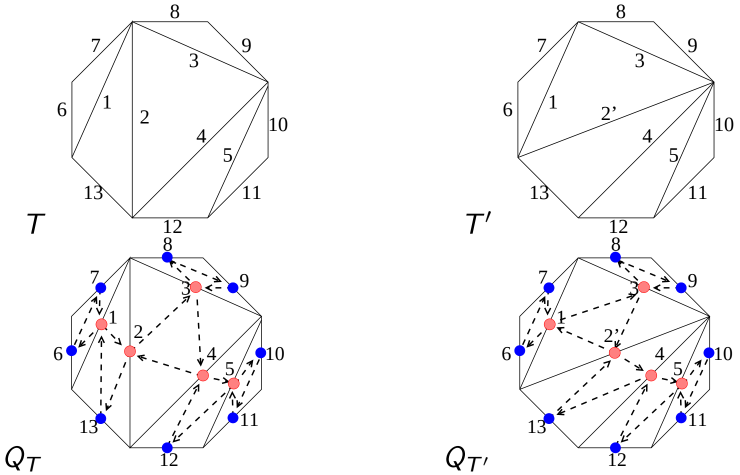


This gives rise to a cluster algebra $\mathcal{A}(Q_T)$, with initial seed $(Q_T, \{x_1, \ldots, x_{2d-3}\})$.

The set of triangulations of a polygon is connected by *flips*.

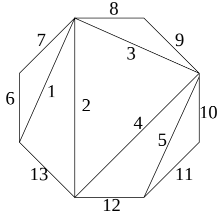


Flips correspond to mutations.

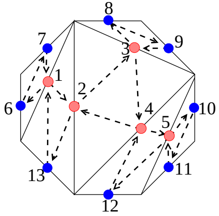


Note that $\mu_2(Q_T) = Q_{T'}$.

Cluster algebras and triangulations of a polygon



T

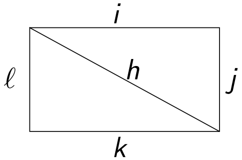


Q_T

Triangulations are connected by flips, and flips $\leftrightarrow$ mutations. Moreover:

$$\begin{aligned} \text{seeds} &\leftrightarrow \text{the triangulations of the polygon} \\ \text{coefficient variables} &\leftrightarrow \text{the } d \text{ sides of the polygon} \\ \text{cluster variables} &\leftrightarrow \text{the } \frac{d(d-3)}{2} \text{ diagonals of the polygon} \end{aligned}$$

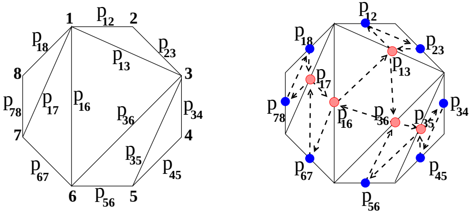
Exchange relation:



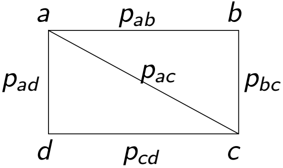
$$x_h x_h' = x_i x_k + x_j x_\ell$$

Cluster algebras and triangulations of a polygon

Relabel the triangulation, and coefficient/cluster variables as follows:



Exchange relation:

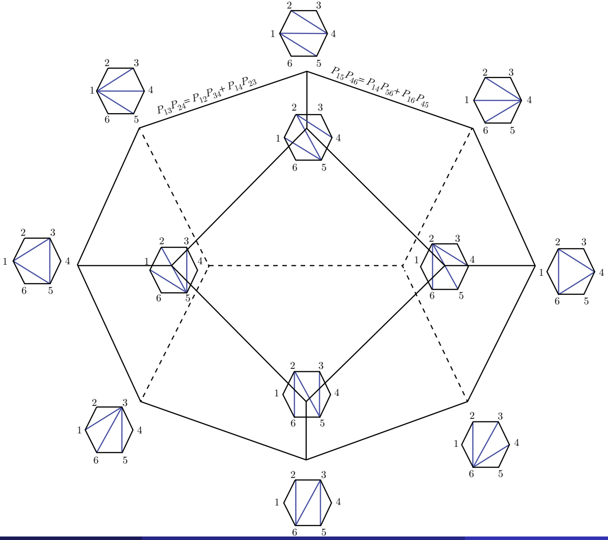


$$p_{ac}p_{bd} = p_{ab}p_{cd} + p_{bc}p_{ad}$$

This identifies our cluster algebra with the coordinate ring of the Grassmannian $\mathbb{C}[Gr_{2,d}]$! Every cluster (triangulation) gives rise to a positivity test for membership in $(Gr_{2,d})_{>0}$. (Why?)

Cluster algebras and triangulations of a polygon

The cluster algebra associated to $\mathbb{C}[Gr_{2,d}]$ can be visualized using the *associahedron*:



Two generalizations of this cluster algebra

Cluster algebra from triangulations of a polygon

||

$\mathbb{C}[Gr_{2,d}]$

Cluster algebra from triangulations of a Riemann surface

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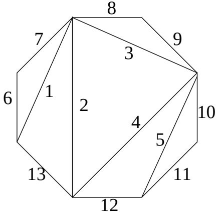
Teichmuller theory

The coordinate ring $\mathbb{C}[Gr_{k,d}]$

First generalization: polygon $\rightsquigarrow$ surface

Recall that given a triangulation of a polygon, we can construct a quiver and an associated cluster algebra.



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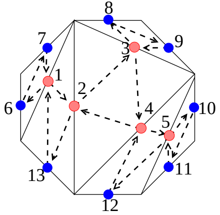


Q_T

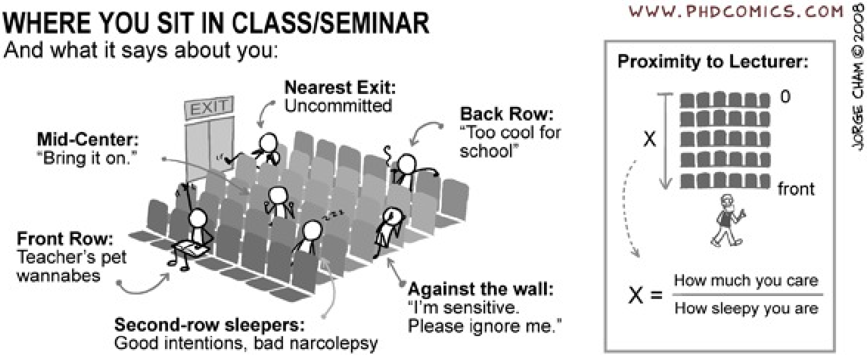
Idea: if we have an oriented surface with some marked points, we can triangulate it, and construct a quiver as before!

Second generalization: triangulation $\rightsquigarrow$ plabic graph

To get positivity tests for $(Gr_{k,n})_{>0}$ for $k > 2$, replace *triangulations* with Postnikov's *(k, n)-plabic graphs.*

To get polytope encoding these positivity tests for $(Gr_{k,n})_{>0}$, replace *associahedron* with *higher associahedra* (Galashin-Postnikov-W.)

BREAK


WHERE YOU SIT IN CLASS/SEMINAR
And what it says about you:
EXIT
Nearest Exit:
Uncommitted
Mid-Center:
"Bring it on."
Back Row:
"Too cool for school"
Front Row:
Teacher's pet wannabes
Second-row sleepers:
Good intentions, bad narcolepsy
Against the wall:
"I'm sensitive. Please ignore me."
WWW.PHDCOMICS.COM
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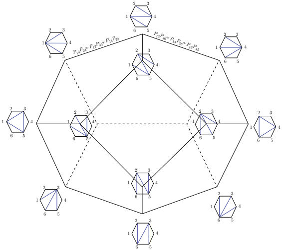
Proximity to Lecturer:
0
X
front
X = How much you care / How sleepy you are

References



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