Moduli Spaces of Tropical Curves
(Part 2)

ICERM Bootcamp

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Sam Payne (UT Austin)

Topology of $\Delta_{g,n}$:

Example: $\Delta_{0,n} \simeq \underbrace{\bigvee_{i=1}^{(n-2)!} S^{n-4}}_{\text{spaces of phylogenetic trees.}}$ (Billera-Holmes-Vogtmann)

Thm: (Chan-Galatius-P, Allcock-Corey-P).

If $g>0$, then the following closed subcomplexes of $\Delta_{g,n}$ are contractible.

- $\Delta^{w}_{g,n} = \{$ tropical curves w/ vertices of positive weight $\}$
- $\Delta^{lw}_{g,n} = \{$ tropical curves w/ loop edges on vertices of positive weight $\}$
- $\Delta^{rep}_{g,n} = \{$ tropical curves w/ repeated markings $\}$

- $\Delta_{g,n}^{br} = \{$ tropical curves w/ bridges $\}$
- $\Delta_{g,n}^{bm} = \{$ tropical curves w/ bridges or multiple edges $\}$

And $\Delta_{g,n}$ is simply connected.

<u>Example</u>: $\Delta_3 = \Delta_3^{bm} \cup \Delta_{K_4}$ $\left(= \Delta_3 \cap M_{K_4}^{trop}\right)$.

 $\Rightarrow$ $\Delta_3 \simeq S\left(S^4 / \underbrace{S_4}_{Aut(K_4)}\right)$

And $S(S^4/S_4) \underbrace{\cong}_{\text{homeomorphism}} S^5$ (Blum Smith-Marques, Lange 2018-9)

Example: $\tilde{H}_*(\Delta_4;\mathbb{Q})=0$ but $\tilde{H}_*(\Delta_4;\mathbb{Z})\neq 0$

(Allcock Corey P–).

Open Problems: homotopy type of Δ_4, integral homology of Δ_5, rational homology of Δ_g for $g \geq 8$ ...

"done" by floating pt calc Willwacher Zivkovic ...

Structure of Tropical Moduli Spaces

$\Delta_{g,n}$ and related moduli spaces (such as the link of A_g^{trop} (Brandt-Bruce-Chan-Melo-Moreland-Wolfe) are symmetric CW-complexes, i.e., filtered spaces

$$X_0 \subseteq X_1 \subseteq \dots \subseteq X$$

where X_n is obtained by attaching quotients B_n / H along $S^{n-1}/H \to X_{n-1}$ for some finite subgps $H < O(n)$.

$\rightsquigarrow$ Cellular homology $C_n(X) \cong \mathbb{Q}^{\left\{ \substack{n\text{-cells such that} \\ H < SO(n)} \right\}}$

For $\Delta_{g,n}$: $C_k(\Delta_{g,n}) \cong \mathbb{Q}\left\{\begin{array}{l}\text{Graphs w/ } k+1 \text{ edges st} \\ \mathrm{Aut}(G) \text{ acts evenly on } E(G)\end{array}\right\}$

differential = signed sum of edge contractions.

Thm (Chan Galatius P)

$$C_*(\Delta_g, \Delta_g^w) \cong G^{(g)}$$

← Kontsevich's graph complex (1993)

additional Lie structure (after $\bigoplus_g / \prod_g$). dgla

Thm (Willwacher 15).

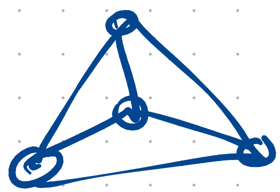
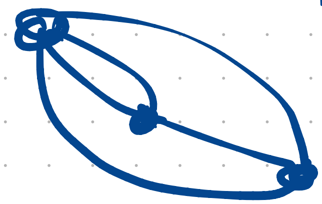
$$\prod_g H^{2g-1}(\Delta_g, \Delta_g^w) \cong \widehat{\mathfrak{grt}_1}$$

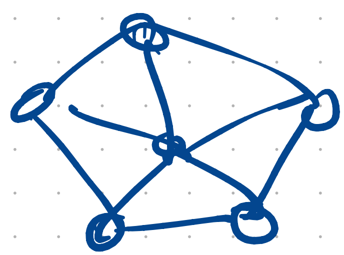
Thm (Brown 12).

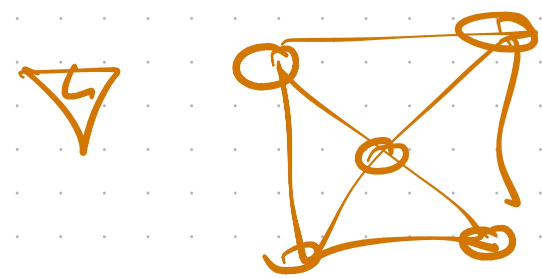
$\mathfrak{grt}_1$ contains a free Lie subalgebra w/ gens $\sigma_3, \sigma_5, \sigma_7, \sigma_9 \ldots$

$$\sigma_g \in H^{2g-1}(\Delta_g)$$

Examples of cycles in $C_{2g-1}(\Delta_g)$

$d\Big($  $\Big) = c \cdot \Big[$  $\Big] \simeq 0$ in $C_{2g-2}(\Delta_g)$

$d\Big($  $\Big) = 0$

 $= 0$

Thm (Willwacher 15)

$$\langle \sigma_g, \omega_g \rangle \neq 0.$$

Open Problems: Study matroid analogue G's)

Study related tropical moduli spaces (e.g. tropical Hurwitz spaces Kannan Li Serpente Yun 2020)