Moduli Spaces of Tropical Curves

ICERM Bootcamp

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Recall: M_g = algebraic variety (scheme, DM stack)

($g \geq 2$) $\quad M_g(k) = \{\text{smooth projective } k\text{-curves of genus } g\}/\cong$

$$M_g(\mathbb{C}) = \mathcal{T}_g \,/\, \mathrm{Mod}(S_g)$$

Teichmüller Space = hyperbolic metrics on S_g

Mapping Class group

$$\boxed{H^*(M_g(\mathbb{C}))} =$$

From this morning (M. Brandt)

K = algebraically closed valued field.

$R \subset K$ valuation ring, k = residue field.

Each $X \in M_g(K)$ has a unique stable model

$$\begin{array}{c}\mathcal{X}\\ \downarrow\\ \operatorname{Spec} R\end{array}$$

$\mathcal{X}_K = X$ and $\omega_{\mathcal{X}}$ = (rel.) ample.

$\rightsquigarrow$ abstract tropical curve

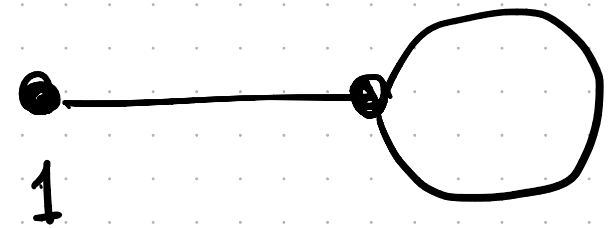
G = dual graph of $\mathcal{X}_k$ $\begin{cases} V(G) = \text{irreducible components} \\ E(G) = \text{nodes} \end{cases}$

$l : E(G) \to \mathbb{R}_{>0}$ "Thickness of node"

thickness of $(xy - f = 0) := \operatorname{val}(f)$

Ex: $g=2$

$\mathfrak{X}_k$



vertices labeled by genus

G

Note: • Ampleness of $\omega_{\mathfrak{X}}$ depends only on G

• $\omega_{\mathfrak{X}}$ ample $\iff$ every vertex of genus 0 has valence at least 3

• Only finitely many such stable graphs for each g.

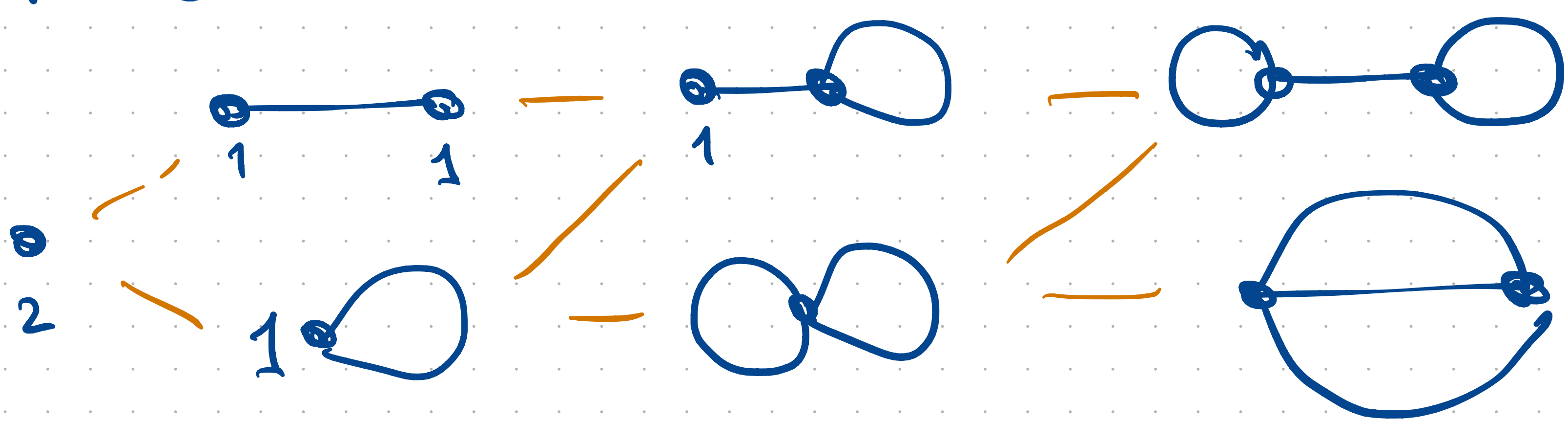
Def: $M_g^{trop} = \{\text{stable tropical curves of genus } g\} / \cong$

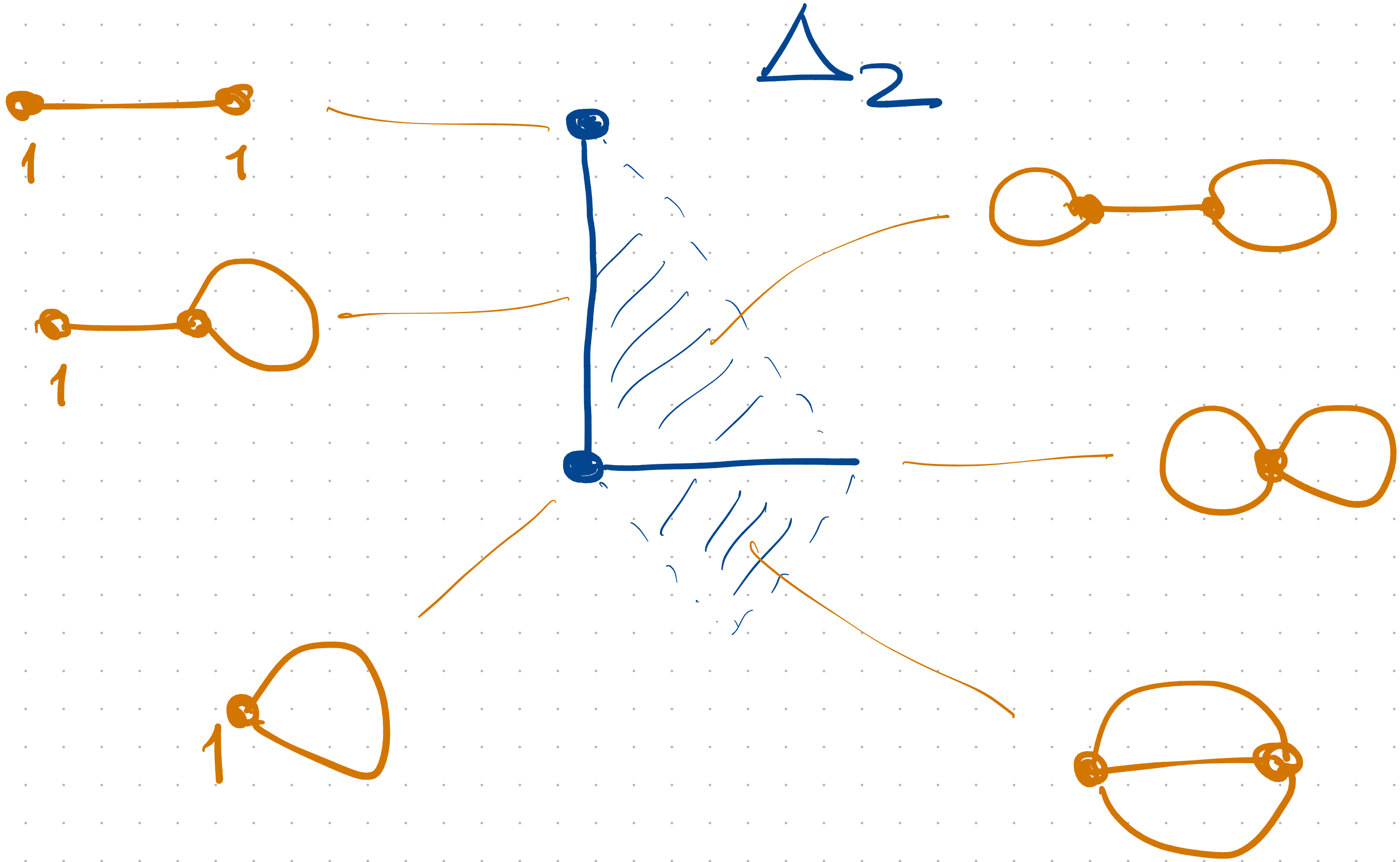
$$= \bigsqcup_G \underbrace{\mathbb{R}_{>0}^{E(G)} / \mathrm{Aut}(G)}_{M_G^{trop}}$$

topology: $M_{G/e}^{trop} \subseteq \overline{M_G^{trop}}$

Note: M_g^{trop} = Cone over $\Delta_g := \{(G,\ell) : \sum_e \ell(e) = 1\}$.

Example: $g=2$





Marked Points: Fix $g, n \geq 0$ such that $2g-2+n > 0$.

$$M_{g,n}(K) = \left\{ \begin{array}{c} \text{sm. proj. } K\text{-curves of genus } g \text{ w/} \\ n \text{ distinct marked pts} \end{array} \right\} \Big/ \cong$$

$(X; p_1, \ldots, p_n)$

Note: $2g-2+n>0 \iff \omega_X(p_1+\cdots+p_n)$ ample.

Stable reduction:

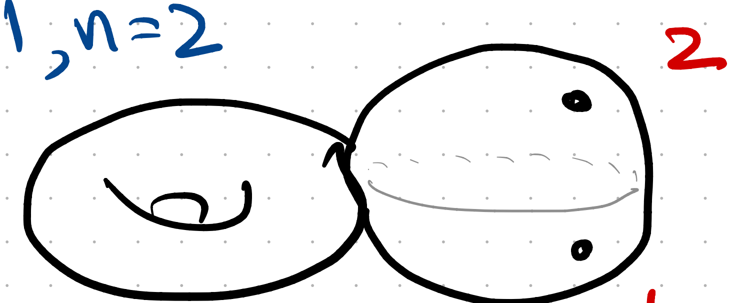
$(X; p_1, \ldots, p_n)$ has a unique stable model

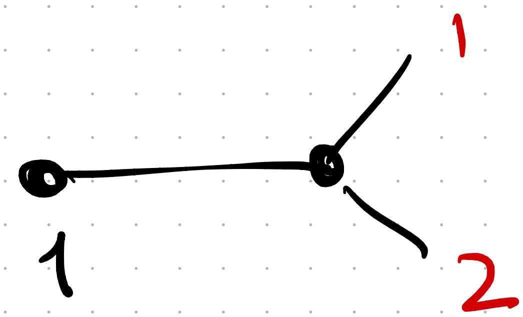
$$\underbrace{(\mathcal{X}; \sigma_1, \ldots, \sigma_n)}_{\text{disjoint sections of } \mathcal{X}^{sm}}$$

- $\mathcal{X}_k = X$ and $\sigma_i = \overline{p_i}$
- $\omega_{\mathcal{X}}(\sigma_1+\cdots+\sigma_n)$ (rel.) ample

Dual graph of $\mathcal{X}_k$ = stable n-marked tropical curve

Ex: $g=1, n=2$





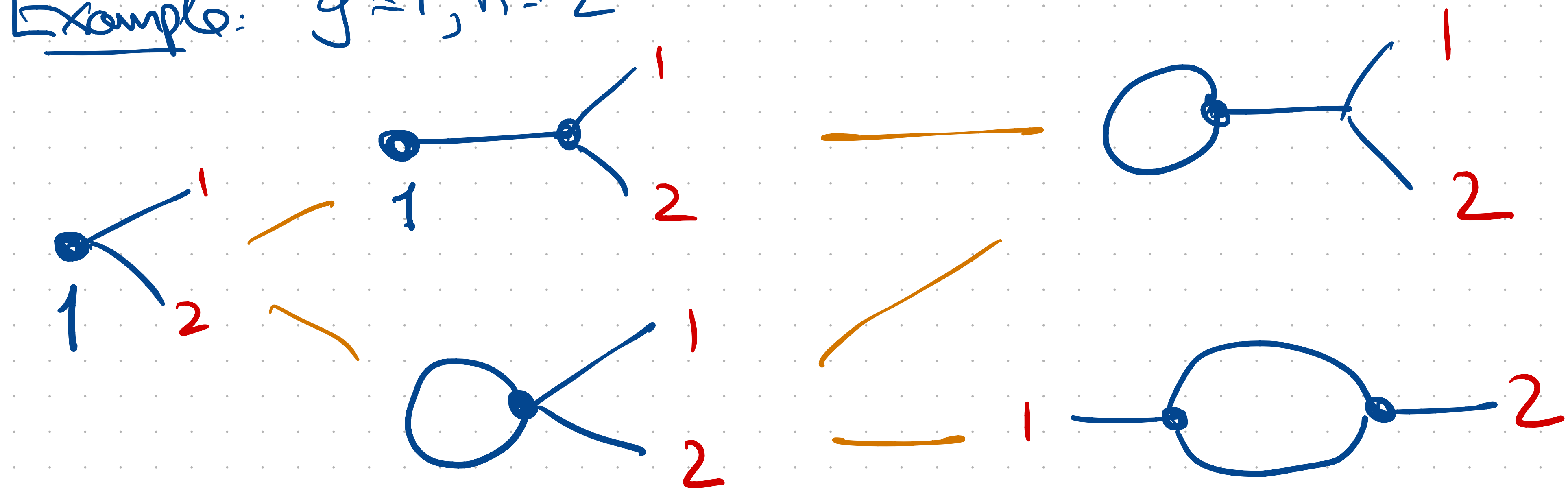
Stable: each vertex of genus 0 has valence ≥ 3

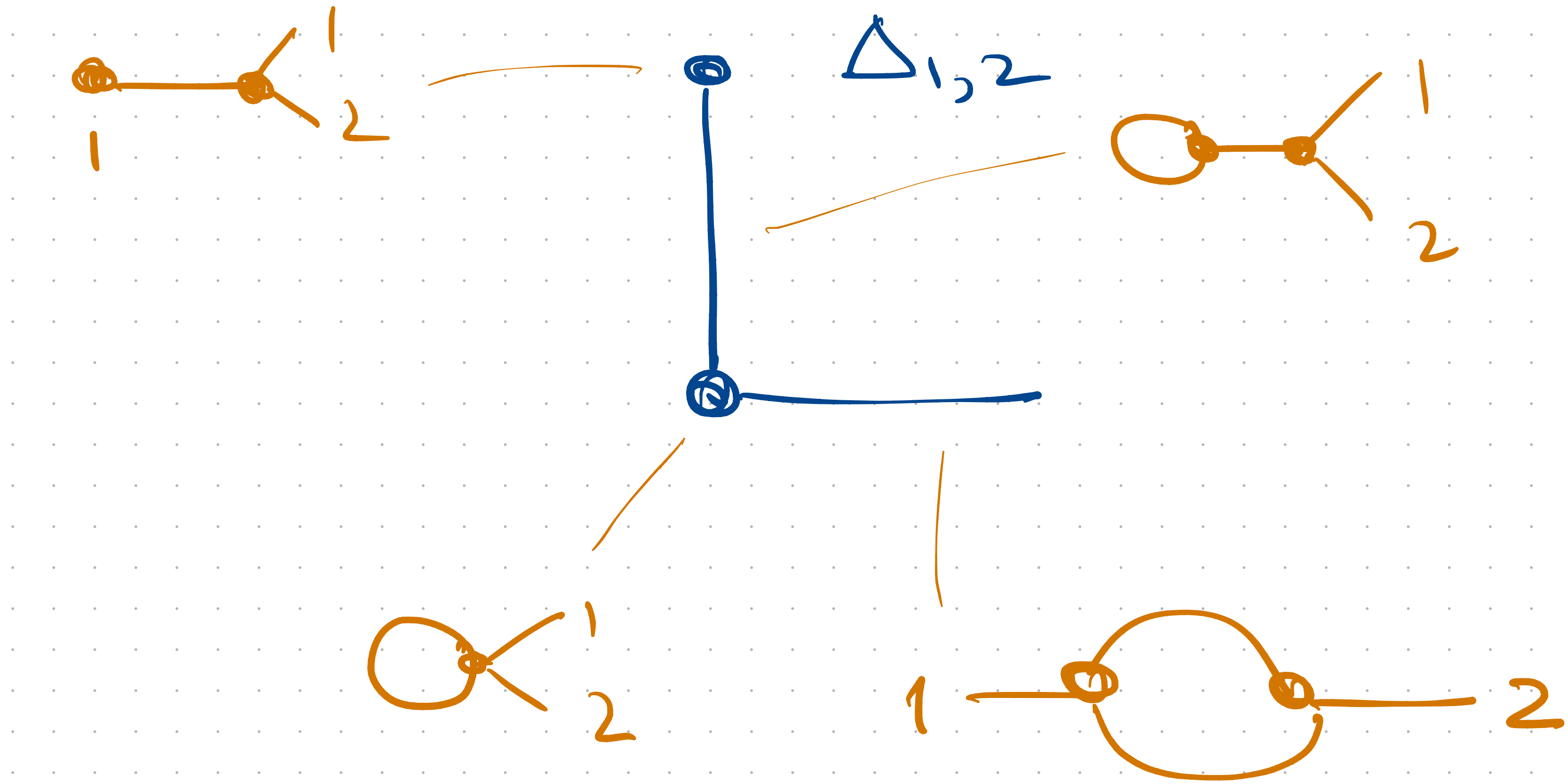
Note: finitely many dual graphs for each g, n.

$$M_{g,n}^{trop} = \{\text{stable } n\text{-marked tropical curves of genus } g\} / \cong$$

$$= \text{cone over } \Delta_{g,n} := \{(G,\ell) : \sum_e \ell(e) = 1\}$$

Example: $g=1, n=2$





<u>A map</u>: $\lambda : M_g(\mathbb{C}) \longrightarrow M_g^{trop}$. (recall: $M_g(\mathbb{C}) = T_g/\mathrm{Mod}(S_g)$)

$X \in T_g$ hyperbolic metric

Fix small $\varepsilon > 0$.

Let G = dual graph of nodal surface obtained by contracting all closed geodesics of length $< \varepsilon$.

$$e \in E(G) \longleftrightarrow \text{geodesic of length } a_e < \varepsilon$$

$$\ell(e) := -\log(a_e / \varepsilon).$$

$$\boxed{\lambda : X \longmapsto (G, \ell)}$$

Observation: λ is proper and hence induces

$$\lambda^* : H_c^*(M_g^{trop}) \longrightarrow H_c^*(M_g(\mathbb{C}))$$

Thm (via Deligne's MHS): $\lambda_{\mathbb{Q}}^*$ is injective

Equivalently (via Poincaré duality)

$$H^{6g-6-k}(\mathcal{M}_g;\mathbb{Q}) \twoheadrightarrow \widetilde{H}_{k-1}(\Delta_g;\mathbb{Q})$$