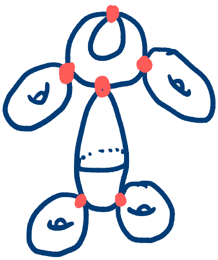
# Introduction to Tropical Geometry at ICERM

February 4 2021

Madeline Brandt

# Overview: 2 friends

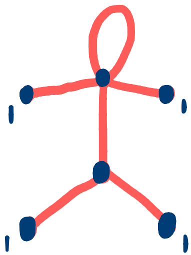
Tropical geometry tells us how to relate these friends:



Algebraic Geometry

Algebraic Varieties

$\{x \in K^n \mid f_1(x) = \cdots = f_\ell(x) = 0\}$



Combinatorics

$\xrightarrow{\text{trop}}$ tropical varieties / polyhedral complexes

**Today:**

Part I: Embedded tropical geometry via curves in the plane

Part II: Abstract tropical geometry & the two friends.

# Geometry over Non-Archimedean fields

Tropical Geometry deals with varieties over Non-Archimedean fields. These fields have a norm that behaves very differently from the Archimedean norm on $\mathbb{C}$.

Definition. $(K, |\cdot|)$ is an Archimedean field if it satisfies the Archimedean Axiom:

for any $x \in K^*$, there is an $n \in \mathbb{N}$ such that $|nx| > 1$.

This axiom feels natural and familiar – but $\mathbb{R}$ and $\mathbb{C}$ are the only complete Archimedean fields (Ostrowski's theorem)

A non-Archimedean field $K$ is one with a norm which fails this axiom.

It comes with a function called the valuation

$$\mathrm{val}_K : K \to \mathbb{R} \cup \{\infty\}$$

$$a \mapsto -\log(|a|) \quad a \neq 0, \qquad 0 \mapsto \infty$$

Example. The trivial valuation on any $K$ is: $0 \mapsto \infty$, $K^* \mapsto 0$

Example. The Puiseux series $\mathbb{C}\{\{t\}\}$ is:

$$\left\{ c(t) = c_1 t^{a_1} + c_2 t^{a_2} + \dots \;\middle|\; \begin{array}{l} c_1 \neq 0, \; c_i \in \mathbb{C} \\ a_i \text{ an } \nearrow \text{ seq. in } \mathbb{Q} \\ \text{w/ common denominator} \end{array} \right\} \cup \{0\}$$

norm: $|c(t)| = (1/e)^{a_1}$

val: $\mathrm{val}_K(c(t)) = -\log(|c(t)|) = -\log((1/e)^{a_1})$

$= -a_1 \log(1/e) = a_1$

* alg. closed *

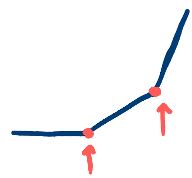
# Embedded tropicalization

How to find the embedded tropicalization of a hypersurface over a non-Archimedean field.

**Definition.** Given a Laurent polynomial

$$f(x) = \sum_{a \in \mathbb{Z}^n} c_a x^a \quad \in \quad K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

its **tropicalization** $\mathrm{trop}(f) : \mathbb{R}^n \to \mathbb{R}$ is



$$\mathrm{trop}(f)(x) = \min_{a \in \mathbb{Z}^n} \left( \mathrm{val}_K(c_a) + a \cdot x \right)$$

Just as we can associate a variety to $f$, which would be a hypersurface in $(K^*)^n$, we will associate a **tropical variety** to $\mathrm{trop}(f)$.

**Definition.** The **tropical hypersurface** $\mathrm{trop}(V(f))$ is the set

$$\{ w \in \mathbb{R}^n \mid \text{the minimum in } \mathrm{trop}(f)(w) \text{ is achieved at least twice} \}$$

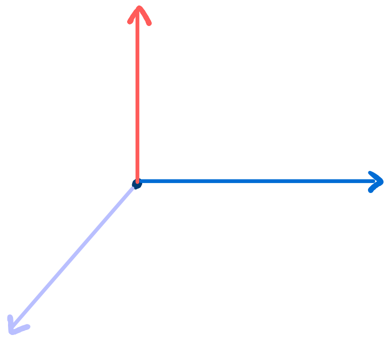
**Example [tropical line].** Let $f = x + y + 1 \in \mathbb{C}\{\{t\}\}[x,y]$.

Then $\mathrm{trop}(f) : \mathbb{R}^2 \to \mathbb{R}$, and

$$\begin{aligned} \mathrm{trop}(f)(w_1, w_2) &= \min\big(\mathrm{val}_K(1) + (1,0)\cdot(w_1,w_2), \\ &\qquad \mathrm{val}_K(1) + (0,1)\cdot(w_1,w_2), \\ &\qquad \mathrm{val}_K(1)\big) \\ &= \min(w_1, w_2, 0) \end{aligned}$$

make 3 cases:

- $w_1$ & 0 are min: $w_1 = 0,\ w_2 \geq 0$
- $w_2$ & 0 are min: $w_2 = 0,\ w_1 \geq 0$
- $w_1$ & $w_2$ are min: $w_1 = w_2,\quad w_1 \leq 0.$



← tropical line

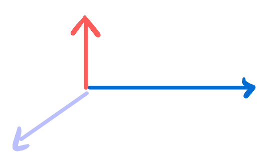
(note: this would be very inefficient if you had many terms)

**Theorem [Kapranov ⇝ Fundamental Thm]**
The set $\mathrm{trop}(V(f))$ is the same as

$$\overline{\{(\mathrm{val}_k(y_1), \dots, \mathrm{val}_k(y_n)) \mid (y_1, \dots, y_n) \in V(f)\}}$$

**Example [tropical line].** Let $f = x + y + 1 \in \mathbb{C}\{\{t\}\}[x,y]$.
$y = -x-1 \Rightarrow$ all solns are $(x, -x-1)$.
Kapranov Theorem $\Rightarrow \mathrm{trop}(V(f)) = \{(\mathrm{val}_k(x), \mathrm{val}_k(-x-1)) \mid x \in k\}$

$\mathrm{val}_k(x) > 0$: $\mathrm{val}_k(-x-1) = 0$
$\mathrm{val}_k(x) = 0$: $\mathrm{val}_k(-x-1) \geq 0$
$\mathrm{val}_k(x) < 0$: $\mathrm{val}_k(-x-1) = \mathrm{val}_k(x)$



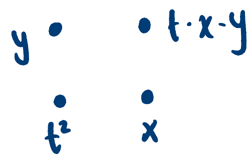
**Remark.** What kind of object is a tropical variety?
A lot can be said about its structure [Structure Theorem]

**Proposition** [Practical method for curves in the plane by hand]

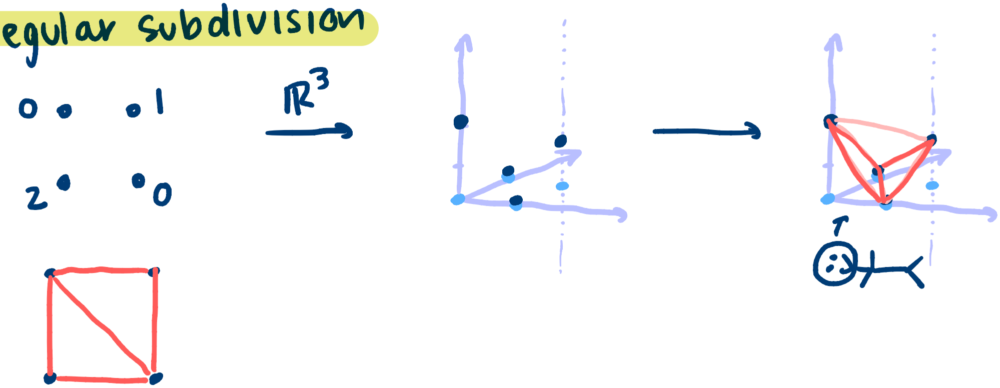
Let $f \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. The tropical hypersurface $\mathrm{trop}(V(f))$ is the (n-1)-skeleton of the polyhedral complex dual to the regular subdivision of the Newton polytope of $f$ induced by the weights $\mathrm{val}(c_u)$

**Example.** Let $f(x,y) = t \cdot x \cdot y + x + y + t^2 \in \mathbb{C}\{\{t\}\}[x,y]$

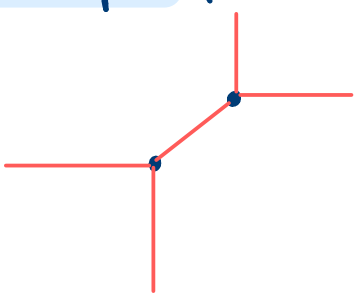
1. Newton Polytope: one lattice point per monomial




2. Regular subdivision

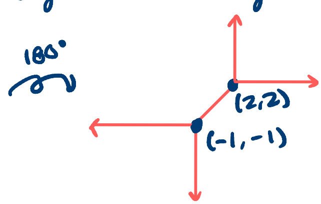



3. dual complex (rotate by 180°)



vertex ⟷ 2-dim cell

edges ⟷ edges




$\mathrm{trop}(f) = \min(2, x, y, 1+x+y)$

# Exercises.

**Problem 1.** Let $k = \mathbb{C}\{\{t\}\}$, and let

$$f(x,y) = t^3 + tx + tx^2 + t^3x^3 + ty + xy + tx^2y + ty^2 + txy^2 + t^3y^3$$

1. Compute $\operatorname{trop}(f)$.
2. Using any method you like compute $\operatorname{trop}(V(f))$.
3. $V(f)$ is an elliptic curve. Every elliptic curve over $\mathbb{C}\{\{t\}\}$ can be re-embedded so that its equation is of the form $y^2 = x^3 + ax + b$ for $a, b \in \mathbb{C}\{\{t\}\}$ (Weierstrass form). What are all the possibilities for tropicalizations of elliptic curves in Weierstrass form?

**Problem 2.** Let $a \in \mathbb{C}\{\{t\}\}^*$ and $b, c \in \mathbb{C}\{\{t\}\}$.

1. Determine $\operatorname{trop}(V(a\cdot x + b))$.
2. Determine $\operatorname{trop}(V(a\cdot x^2 + bx + c))$.

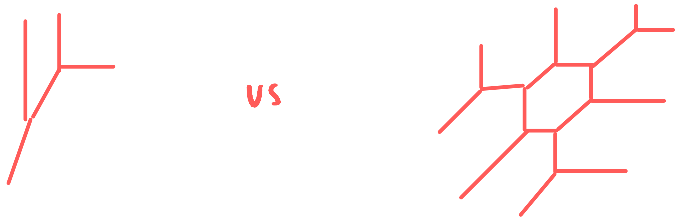
**Problem 3.** How many combinatorial types of tropical quadratic curves are there? ie, tropicalizations of

$$0 = ax^2 + bx + c + dy + exy + fy^2$$

for $a, \dots, f \in \mathbb{C}\{\{t\}\}^*$.

# Abstract Tropicalization

From exercise 1 in the problem session, you saw that a curve can have different embedded tropicalizations depending on the embedding:

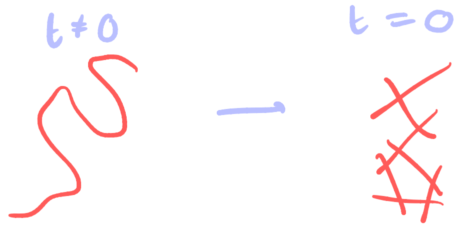


**Theorem [Chan-Sturmfels]** Every elliptic curve with $\mathrm{val}_K(j) < 0$ has an embedding such that the embedded tropicalization is a honeycomb.

**Question.** How do we associate an "intrinsic" tropical object to a curve?

# Abstract Tropicalization

A curve over $\mathbb{C}\{\{t\}\}$ can be thought of as a family of curves depending on a parameter $t$.



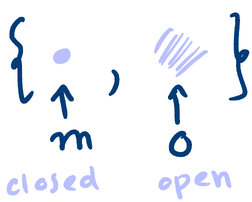

Informally we can think of this parameter as "going to zero", and when $t=0$, we observe some possibly singular behavior.

## Notation.

Let $R = \{ f \in K \mid \mathrm{val}_K(f) \geq 0 \}$.

This is a local ring with unique maximal ideal $\mathfrak{m} = \{ f \in K \mid \mathrm{val}_K(f) > 0 \}$.

Spec(R) is a topological space with 2 points:



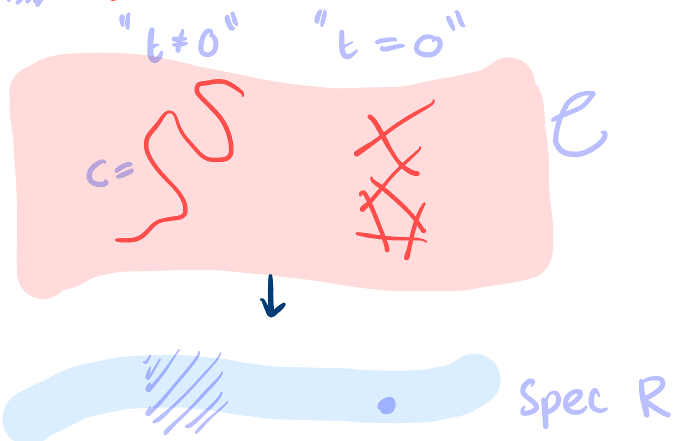
# Models of Curves

More formally, we need models of curves.

Let $K$ be complete w.r.t $\mathrm{Val}_K$ (e.g. the completion of $\mathbb{C}\{t\}$)

Suppose $C$ is a smooth and proper curve over $K$.

Definition. A model $\mathcal{C}$ of $C$ over $K$ is a proper and flat scheme over $R$ whose fiber over ▨ (generic fiber) is isomorphic to $C$.




$\mathcal{C}$ is called semistable if the fiber over • (special fiber) is reduced, has at worst nodal singularities, and every rational component has at least 2 singular points.

✱ combinatorial ✱

**Remark.** By the semistable reduction theorem we are always guaranteed that a semistable model for C exists.

The semistable reduction theorem guarantees that we can put curves into a combinatorially tractible form.

You can think of this as a "good" embedding from the perspective of tropical geometry.

**Example.** Consider $y^2 = x^3 + x^2 + t^4$ over $\mathbb{C}\{\{t\}\}$.

smooth elliptic curve

$y^2 = (x+1)x^2$

Spec R

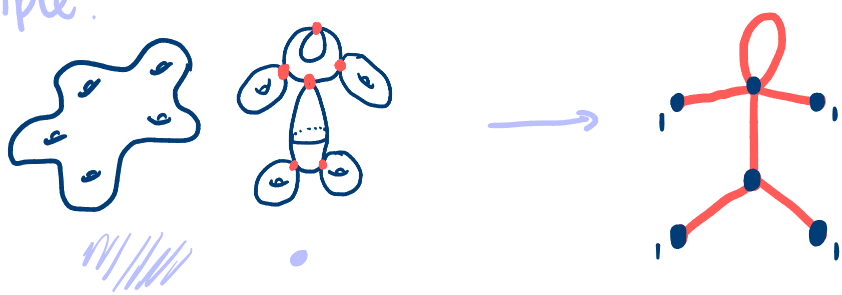
# Dual Graphs

Let $C$ be a curve over $K$, and let $\mathcal{C}$ be a semistable model.

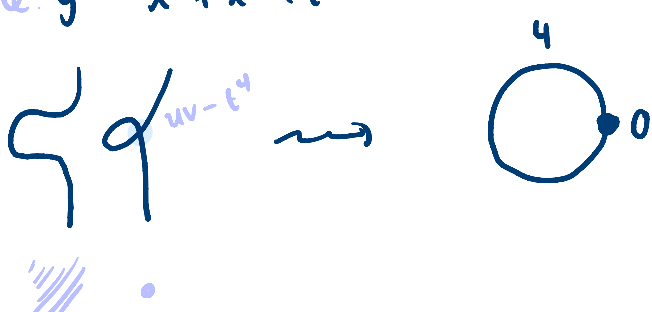
The abstract tropicalization $\Gamma$ of $\mathcal{C}$ is a metric graph with:

- vertices $\longleftrightarrow$ irreducible components of the special fiber
- edges $\longleftrightarrow$ nodes of the special fiber
- vertex weights $\longleftrightarrow$ genus of the component
- edge lengths $\longleftrightarrow$ deformation parameter at each node
  (locally: $xy - f$ for $f \in R$. $\mathrm{val}_K(f)$ is length)

Example.




Example. $y^2 = x^3 + x^2 + t^4$



# minimal skeletons

A tuple $(G=(V,E), w, \ell)$ is called a tropical curve.

$w: V \to \mathbb{N}$ $\quad$ $\ell: E \to \mathbb{R}_{\geq 0}$

The genus of a tropical curve is $\sum_{v \in V} w(v) + |E| - |V| + 1$

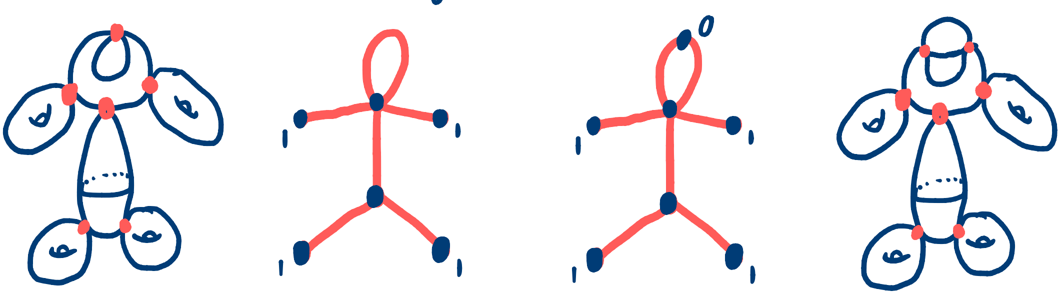
We say two tropical curves $(G, w, \ell)$ are isomorphic if one can be obtained from the other by

- graph automorphisms 
- contracting weight 0 leaves 

- "erasing" valence 2 weight 0 vertices 
- contracting length 0 edges. 


Remark. Different semistable models for a curve $C$ will have isomorphic tropical curves. (ie, "tropical curve of $C$" is well-defined).

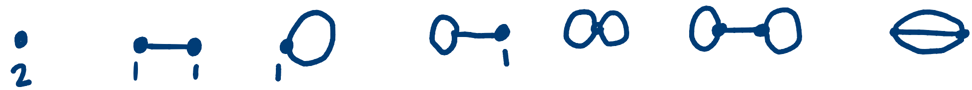
Example. The following tropical curves are isomorphic



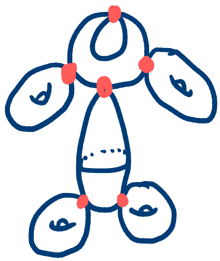
**Proposition.** Every tropical curve of genus $\geq 2$ has a **minimal skeleton**, which is a $(G=(V,E), \ell, w)$ with

- no vertices of weight 0 and degree $\leq 2$
- no edges of length 0

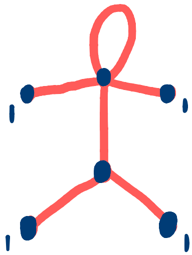
**Example.** Here are all combinatorial types (i.e., forget $\ell$) of tropical curves of genus 2:



# 2 friends

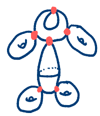
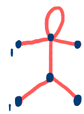


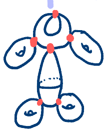
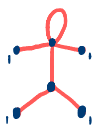
Algebraic Geometry



Combinatorics

## How are the friends related?

- the abstract tropicalization of  is .

- "good" or faithful embedded tropicalizations of  contain .

**Remark.** Today I focused on curves... but this can be done for higher dimensional varieties!!


Thank You!