Affine Stanley functions [Thomas Lam 2005]

A word $(a_1, a_2, \ldots, a_\ell)$ with $a_j \in \tilde{I} = \mathbb{Z}/n\mathbb{Z}$ is *cyclically decreasing* (c.d.) if

- The a_j are distinct
- If i and $i+1$ are both present then $i+1$ occurs first.

Say $w \in \tilde{S}_n$ is c.d. if it has a c.d. reduced word.

Example For $n = 4$ $s_0 s_3$ is c.d. but $s_1 s_0 s_2$ is not.

Remark A c.d. element of $\tilde{S}_n$ has length at most $n-1$.
For $w \in \tilde{S}_n$ define the **affine Stanley function** $\tilde{F}_w$ to be the series

$$\tilde{F}_w = \sum_{w = v^{(1)} v^{(2)} \cdots v^{(r)}} x_1^{\ell(v^{(1)})} x_2^{\ell(v^{(2)})} \cdots x_r^{\ell(v^{(r)})}$$

where r varies freely and $w = v^{(1)} \cdots v^{(r)}$ is a length-additive factorization with $v^{(j)}$ c.d. ($v^{(j)}$ can be the identity).

Affine Stanleys are symmetric functions

Proposition

$\tilde{F}_w \in \Lambda$.

For $w \in \tilde{S}_n^0$ we call $\tilde{F}_w$ the affine Schur function.
Same as dual k-Schur function [Lapointe, Morse 2005]

Remark

There is a bijection from $\tilde{S}_n^0$ to the set of $(n-1)$-bounded partitions (λ such that $\lambda_1 \leq n-1$). In the literature bounded partitions are often used instead of $\tilde{S}_n^0$.

Proposition

For $w \in \tilde{S}_n^0$, $\tilde{\mathfrak{S}}_w = \tilde{F}_w$. In particular $\{\tilde{F}_w \mid w \in \tilde{S}_n^0\}$ is a basis of $\Lambda^n(x_-)$.

Affine Stanley-to-Schur coefficients are homology Schubert structure constants

For $x \in \tilde{S}_n$ and $u \in \tilde{S}_n^0$ define $j_u^x \in \mathbb{Q}$ by

$$\tilde{F}_x = \sum_{u \in \tilde{S}_n^0} j_u^x \tilde{F}_u.$$

Theorem (Peterson 1997)

$d_{uv}^w = j_u^{wv^{-1}}$.

Proof uses Peterson's realization of $H_*(\tilde{\mathrm{Gr}})$ as a commutative subalgebra of a small-torus affine nilHecke algebra and characterization of the Schubert basis (Peterson's j-basis).

$H^*(\tilde{\mathrm{Fl}})$ as $H^*(\tilde{\mathrm{Gr}})$-Hopf comodule

Define the map $\Delta : H^*(\tilde{\mathrm{Fl}}) \to H^*(\tilde{\mathrm{Gr}}) \otimes H^*(\tilde{\mathrm{Fl}})$ by

$$\begin{array}{c} H^*(\tilde{\mathrm{Fl}}) \\ \big\downarrow \boldsymbol{\Phi} \\ H^*(\tilde{\mathrm{Gr}}) \otimes H^*(\mathrm{Fl}) \\ \big\downarrow \Delta \otimes \mathrm{id}_{H^*(\mathrm{Fl})} \\ H^*(\tilde{\mathrm{Gr}}) \otimes H^*(\tilde{\mathrm{Gr}}) \otimes H^*(\mathrm{Fl}) \\ \big\downarrow \mathrm{id}_{H^*(\tilde{\mathrm{Gr}})} \otimes \boldsymbol{\Phi}^{-1} \\ H^*(\tilde{\mathrm{Gr}}) \otimes H^*(\tilde{\mathrm{Fl}}) \end{array}$$

comultiply on the affine Grassmannian factor

Coproduct formula for affine Schubert polynomials

Theorem (Lam, Lee, Shimozono 2015)

For $w \in \tilde{S}_n$

$$\tilde{\mathfrak{S}}_w = \sum_{\substack{uv=w \\ \ell(u)+\ell(v)=\ell(w) \\ v\in S_n}} \tilde{F}_u(x_-) \otimes \mathfrak{S}_v \in H^*(\tilde{\mathrm{Gr}}) \otimes H^*(\mathrm{Fl}_n)$$

$$\Delta(\tilde{\mathfrak{S}}_w) = \sum_{\substack{uv=w \\ \ell(u)+\ell(v)=\ell(w)}} \tilde{F}_u(x_-) \otimes \tilde{\mathfrak{S}}_v \in H^*(\tilde{\mathrm{Gr}}) \otimes H^*(\tilde{\mathrm{Fl}})$$

k-Schur functions $\tilde{\mathfrak{S}}_w^\vee$ via homology k-Pieri rule

Let $r_p = s_{p-1}s_{p-2}\cdots s_1 s_0 \in \tilde{S}_n^0$ for $1 \le p \le n-1$.

Fact: $\tilde{\mathfrak{S}}_{r_p}^\vee = h_p(x_-) =$ sum of all monomials in x_- of degree p

Theorem (Lapointe, Morse 2007)

$\{\tilde{\mathfrak{S}}_w^\vee \mid w \in \tilde{S}_n^0\}$ is the unique basis such that for all $v \in \tilde{S}_n^0$ and $1 \le p \le n-1$

$$h_p \tilde{\mathfrak{S}}_v^\vee = \sum_{w \in \tilde{S}_n^0} \tilde{\mathfrak{S}}_w^\vee$$

for $w \in \tilde{S}_n^0$ such that wv^{-1} is c. d. of length p.

- Change of basis in Λ_n from $\{h_\lambda \mid \lambda_1 \le n-1\}$ to $\{\tilde{\mathfrak{S}}_w^\vee \mid w \in \tilde{S}_n^0\}$ is unitriangular.
- Change of basis in Λ^n from $\{m_\lambda \mid \lambda_1 \le n-1\}$ to $\{\tilde{\mathfrak{S}}_w \mid w \in \tilde{S}_n^0\}$ is unitriangular.

using bijection $\{\lambda \in \mathbb{Y} \mid \lambda_1 \le n-1\} \cong \tilde{S}_n^0$

Theorem (Lam, 2008)

- *Under the isomorphism* $\Lambda_n \cong H_*(\tilde{\mathrm{Gr}})$, *the* k*-Schur basis* $\{\tilde{\mathfrak{S}}_w^\vee \mid w \in \tilde{S}_n^0\}$ *maps to the Schubert homology basis.*
- *Under the isomorphism* $\Lambda^n \cong H^*(\tilde{\mathrm{Gr}})$, *the affine Schur (dual* k*-Schur) basis* $\{\tilde{S}_w \mid w \in \tilde{S}_n^0\}$ *maps to the cohomology Schubert basis.*

Conjecture (Lascoux, Lapointe, Morse 2004)

$\tilde{\mathfrak{S}}_w^\vee$ *is Schur positive.*

Theorem (Blasiak, Morse, Pun, Summers 2018)

Explicit Schur expansion of $\tilde{\mathfrak{S}}_w^\vee$.

Directions to go

- T-Equivariant Cohomology
  - double version of affine Schubert exists [LLS]
  - double version of k-Schur exists [Lam, Shimozono] but needs a lot of help.
- (Equivariant) K-theory
  - (double) affine Grothendieck exist in very different forms [LLS 2020] [Kashiwara, Shimozono 2009]
  - Double version of K-analogue of k-Schur???
- Maximal torus equivariance: Rees ring construction [Yun]
- Other types
- Back-stable Schubert calculus (mostly easier than affine Schubert calculus)

References

- Best reference for beginners: k-Schur book [LLMSSZ] Includes: Affine Stanley functions [Lam1]; dual k-Schur functions [LM]; k-Schur functions in affine Grassmannian Schubert calculus [Lam2] [LM2]; Quantum Equals Affine Theorem [P] [LS]
- Hopf structure on cohomology of affine flags and the coproduct formula [LLS]
- Affine Schubert polynomials [Lee]
- Kostant's change of variables: quantum Schubert polynomials to k-Schur functions [LS2]
- Geometry of affine flag ind-varieties and affine Grassmannian [Ku] [Ku2] [KK] [PS]
- Geometry of thick affine flag schemes: [Ka] [KS]

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